THE
MATHEMATICAL GAZETTE

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EUCLID.

THE Mathematical Association was founded in 1871 as the Association for the Reform of Geometrical Teaching, and the reformers aimed, primarily and almost to the exclusion of any other reform, at the abolition of the use of Euclid's *Elements* as a textbook for young children. The Association may therefore fairly claim the *Elements* as an ancestor, even if, as Professor Neville remarks, the line of descent is illegitimate. Thus in beginning a series of frontispieces, on Professor Chapman's felicitous suggestion, to illustrate noteworthy mathematicians and their works, there is a clear case for making the first of the series Euclidean.

The plate chosen reproduces the title-page of the first printed Euclid in Greek, the *editio princeps* of Simon Grynaeus, published at Basel in 1533. The first printed Euclid in translation was Ratdolt's edition, Venice, 1482. It is perhaps unfortunate that Grynaeus' edition was based on two sixteenth-century manuscripts of poor quality; but it was not until the beginning of the nineteenth century that the great Vatican manuscript was discovered by Peyrard. For further details concerning the Basel Euclid, the reader may be referred to Heath's edition of the *Elements*, vol. i, p. 100.

Thanks to the generosity of Sir James Jeans, the Library of the Mathematical Association possesses a copy of this edition, purchased from the Library of the Royal Astronomical Society. For the present photograph, we are indebted to the British Museum Library and Photographic Department.

THE MATHEMATICAL GAZETTE: OUR HISTORY AND AIMS.*

By T. A. A. Broadbent.

Many members of the Association know the *Gazette* only in its hampered and restricted war-time form. The time is therefore opportune for some remarks on the pre-war *Gazette* and on what may reasonably be expected from the post-war *Gazette*. The history of the *Gazette*, however, cannot be fully understood without some reference to the history of the Association; such reference need be only brief, since a full account is available in the admirable Presidential Address by Mr. A. W. Siddons (XX, No. 237, February, 1936).

The Association owes its existence to one man, though he died some 2000 odd years before its birth. Euclid wrote (among other things) a masterpiece, *The Elements of Geometry*, and he wrote it for the cream of the adults of a race marked by a peculiar, perhaps a unique characteristic: the Greeks liked thinking. They delighted in abstract logical thought for its own sake, and admired masters of geometry as we admire a Hobbs or a Verity. The *Elements*, with its orderly system, its brilliant logical triumphs in the theory of parallels and the theory of proportion (for the more Euclid's treatment is examined, the more impressive it becomes, in spite of gaps which later work has revealed), was a peak of Greek culture. How quite it came to be inferred from this that the book was one pre-eminently suited for the education of small boys in Victorian England, how it came to pass that the small boy learned his Euclid as he learned his list of the Kings of Israel and Judah, how it became accepted as incontrovertible that immense mental and spiritual benefits must result from these exercises, forms a curious chapter in the history of education in this country. Even to-day one can find teachers who believe that Euclid is the best thing in the world for the training of small children. It would perhaps be fairer to compare Euclid with Rabelais—both are masters, but neither is really suitable for young children. By the middle of the nineteenth century, some few daring heretics among our teachers were beginning to see this, and even beginning to say it; and these heretics, meeting at Rugby in 1870, founded an Association for the Reform of Geometrical Teaching, and thereby struck the first blow in the revolution in the teaching of school mathematics, a revolution still going on. Of course, the reformers made mistakes; they were bigoted, they were inclined to refuse any virtue to Euclid, their substitutes for him were far from perfect. They worked under great difficulties; they had opposed to them the vested interests and the enormous inertia of the existing system, they had to writhe under the wickedly witty shafts of Lewis Carroll, who satirised them as "The Association for the Improvement of Things in General" and delighted in exposing the sad lack of logic and of common sense in some of the less happy "substitutes for Euclid". Even the change to a milder title by the replacement of the word "Reform" by "Improvement" did little to help the infant Association, which for many years seemed to make only slow progress. The major battle was not won till the first decade of this century, when the Universities of Oxford and Cambridge began to give ground under an attack led by Professor Forsyth, Professor Godfrey and Mr. Siddons.

But while these leaders are honoured, it must not be forgotten that if they were the spearhead of the assault, the shaft was the Association itself, and its strength was mainly due to the solid work of the officers and veterans of the

* The substance of a paper read at the Annual Meeting of the Mathematical Association, 25th April, 1946. At the meeting, an apology was offered, which should perhaps be repeated here, for the somewhat egotistical character of the paper; but in the nature of the case it is impossible to be completely impersonal.

Council: Levett, Pendlebury, Hill, Langley, Macaulay, Greenstreet, J. J. Milne, Alfred Lodge, all of them remarkable men. If Forsyth, Godfrey, Siddons are household words, so too is Pendlebury (Hon. Secretary 1886–1936), a "best" among "best sellers"; one of his cherished possessions was a copy of the *Cornhill*, in which there is an article deducing the tastes, habits and character of C. Pendlebury from a consideration of the examples in his world-famous *Arithmetic*. Then there was F. W. Hill (Hon. Treasurer 1900–1935), slow, cautious, chary of speech, giving little suggestion that in his youth he had been the hero of one of the most tragic of Alpine climbing accidents; and Alfred Lodge, with the magnificent Lodge head and all the Lodge kindness and charm. Two of the men mentioned above are of special interest in the present connection: E. M. Langley and W. J. Greenstreet. It was Langley's suggestion that the Association should publish a journal, and he was its first editor. The idea came at a most suitable time, in the middle of the 1890's. The Association had been in existence for a quarter of a century, and though growing steadily if slowly, it did not appear to have met so far with any striking success.* Its principles were becoming known, and were being here and there accepted, its Annual Reports contained some most admirable articles, but the goal must have seemed still far away, and it is possible that members may have felt that they gained very little immediate benefit from their small subscription. So Langley's idea of a quarterly journal devoted to elementary mathematics was well-timed, and in April 1894 the first *Gazette* appeared, a thin quarto very different in appearance from the stout octavos of recent years. The general programme, too, shows how the school curriculum has altered since 1894. Then, questions concerning the differential and integral calculus, though not absolutely banned, were not regarded as being properly within the scope of school mathematics. The first six numbers, in quarto, include, for example, an article by Langley on the eccentric circle of Boscovich, one by Alfred Lodge on arithmetic, notes on geometrical conics and on algebra by one—perhaps the only one—of the contributors to the first volume still with us, P. J. Heawood, and a typical article, on Herbart's view of the place of mathematics in education, by one whose name was soon to be linked with that of the *Gazette* till the end of his life, W. J. Greenstreet.

Like other young things, the *Gazette* had its teething troubles, particularly concerning the editorial arrangements. Nos. 1–6 were edited by Langley, 7–11 by F. S. Macaulay, 12–15 by Macaulay, Hill and Greenstreet. Macaulay's was a temporary editorship, undertaken only until a permanent editor could be secured; of this period, Macaulay says: "I do not remember how it came about, but very soon I was relying on Greenstreet for untiring help and advice." In July, 1897, Greenstreet wrote that he would help in any way the Council thought advisable; and a year later, under pressure from Macaulay and Lloyd Tanner, he became sole editor, from No. 16, February 1899 to No. 208, July 1930. The covers refer to the editorial cooperation of Macaulay, Lloyd Tanner, and Professor (now Sir Edmund) Whittaker, but though these eminent and busy men were always willing and ready to assist in the work, they themselves would probably have been the first to admit that, for rather more than 30 years, the *Gazette* was Greenstreet. Those who knew Greenstreet personally will know that it is not easy to convey in words the impression of his personality: an amazing, an encyclopaedic knowledge of elementary mathematics, never ostentatiously displayed but unfailingly at your service,

* "There is a society for the Improvement of Geometrical Teaching. I have no knowledge of its work; but as to the need of improvement there can be no question whilst the reign of Euclid continues.... Euclid for children is barbarous." O. Heaviside, *Electromagnetic Theory*, I, (1893), p. 148.

given shrewdly with a kindly wit and a genial humour, seemingly symbolised by the wise and unquenchably merry twinkle in his eye. As Johnson said of Burke, had you but met him for five minutes while sheltering from a shower, you would have known him for an unusual man. His life-work for the *Gazette* will surely not soon be forgotten.

Under Greenstreet's editorship, the life of the *Gazette* falls into two fairly clearly marked sections, divided from one another by that unhappy period 1914–1918. The earlier period, 1900–1914, was perhaps the golden age of the *Gazette* ; if there seems to be some slight falling off in the years after 1918, this is to judge by a high standard, since according to that erudite American critic, Professor R. C. Archibald, under Greenstreet's editorship the *Gazette* stood without a peer among the minor mathematical journals of the world. To give but one instance, there are many reviews, notes and articles by one who was then a young Cambridge don, now the unchallenged leader of pure mathematics in this country, Professor G. H. Hardy. If a mild criticism may be offered, it is that of late years Professor Hardy has tended more and more to insist that problems in the teaching of mathematics are not only of slight importance in comparison with problems of creative mathematics—which indeed they are—but are in themselves insignificant and unimportant. Perhaps his views were changed by seeing how tragically Ramanujan was obstructed and handicapped by stupid teachers ; but members of this Association may continue to believe that as long as we look for sound creative mathematical work in this country, so we must devote our best efforts to the teaching of those who will be the creative workers of the next generation, and may feel that in this we have the support of the Hardy of 1900–1914. In those years, the notes, articles and particularly the reviews which he wrote for the *Gazette* were its outstanding features. The reviews ought still to be read, even though the books which called them forth may have been forgotten. Discriminating but vigorous both in praise and blame, exposing with merciless clarity the slipshod and the careless, those reviews must have contributed very considerably to the marked improvement in English school textbooks which is evident over the last forty years. It is easy, when reviewing, to be bitterly hostile or vaguely approving ; the effect is, however, unmistakable when a critic whose authority cannot be denied, writes : " The defect of this book is that there is nothing in it " ; or of another : " When I began to read this book, I was not acquainted with any of Prof. Kowalewski's writings, but I had not spent an hour over it before I went out and ordered his *Grundzüge* . . . of which this is a continuation." But if Hardy's contributions were outstanding, there were others of almost equal value, and to turn over the pages of those early volumes is to discover readily why they may be considered to belong to the golden age of the *Gazette*.

If the period 1918–1930 shows a less bright and vigorous *Gazette*, as far as this is not an inaccurate judgement, it may be worth suggesting one or two possible reasons. This was the time which saw the growth of the Association's policy of large reports. Till 1923, the teaching reports had been small pamphlets, but the appearance in that year of the famous first *Report on the Teaching of Geometry in Schools* marked a new stage, and it is not impossible that the production of reports of 80 or 100 pages, engaging the energies, over a period of months or of years, of many of our best-known teachers, drew away from the *Gazette* some of the material which it might otherwise have published. This is not to deplore the development of the report policy, but merely to suggest how it may have reacted on the conduct of the *Gazette*. Then, too, in the later years, Greenstreet, though never depressed or complaining, was a tired and an ailing man. But, throughout, he retained the ability to turn a review into a brilliant little article of permanent interest,

however ephemeral the book which prompted it. This is particularly marked in those instances where he gave free rein to the fullness of his knowledge, historical and bibliographical, of some half-forgotten incident or personage. For instance, there are two valuable little books by Macfarlane, *Ten British Mathematicians of the Nineteenth Century*, and *Ten British Physicists . . .*; to see the difference between the treatment of such topics by a competent craftsman and by a real master of the difficult art of handling biographical detail, it is enough to turn up the long review articles on these books which Greenstreet wrote in Vols. IX, X. If the *Gazette* survives to celebrate its 300th number in 1948, perhaps a fitting tribute to Greenstreet would be the reprinting of some examples of his mastery.

It would be tasteless and unwise to attempt to discuss here the merits and demerits of the *Gazette* from 1930 onwards. But some fairly general observations may be made on conduct and policy, on aspirations rather than on achievements. First, however, some words of thanks must be offered ; not all who have helped the *Gazette* can be named here, but some names must be mentioned. First, that of our Librarian, Professor E. H. Neville, of the University of Reading, whose junior colleague I was till 1935. It was he, I believe, who suggested to the Council that they might entrust me with the *Gazette*, and since at that time I knew only enough of editorial routine to realise my own ignorance, his help and counsel was not only welcome, it was vital. And I can still rely, as can all the other officers of the Association, on his critical judgement and his willingness to sacrifice his scanty leisure for the good of the Association. To the Secretaries, in their thankless tasks, Mr. Pendlebury and Miss Punnett first, Mr. Parsons and Mrs. Williams in recent years, no request for assistance has ever been made in vain. During the war years, the motto of the Secretaries and the Treasurer was, so it seems, "Whatever happens, the *Gazette* must go on." The functions of a Treasurer are, of course, usually to say "No" consistently and relentlessly to all calls for extra expenditure. Mr. Hill and Mr. Snell may have been harsh with other sinners, but to the *Gazette* they have always been delightfully lenient. Mr. Snell has steadily and deliberately encouraged the policy of spending more and yet more money on the *Gazette*. The yearly volumes increased in size from 320 pages in 1932 to 528 pages in 1938, and the 1939 volume would have contained about 600 pages had it not been for the outbreak of war. The balance sheets show how large a proportion of the Association's income is spent on the *Gazette*. But if expenditure on the *Gazette* is to be increased, the members of the Association must themselves make it possible for the Treasurer to sanction such an increase ; our membership is good, but not nearly good enough, and every member can help. Individual members can bully their backward colleagues. Branches can organise local drives and convert associates into full members. A membership of 2500 is not an unduly high target and we should not rest easy till we have passed that figure.

One more name : on the last page of each issue of the *Gazette* there is a notice in very small type, giving the name of the printers, Messrs. Robert MacLehose & Co., of the University of Glasgow Press. Their part in the work is all-important, and is so uniformly excellent that readers perhaps do not always realise how good it is, comparing favourably with any mathematical printing of its kind. Contributors who feel pleased on seeing the neat fashion in which their work has been set up may not always realise that the result is often due, partly to some discreet editing and re-writing, but mainly to the skill and care of Messrs. MacLehose. Perhaps no one but the editor can fully appreciate the real interest taken by Messrs. MacLehose in the production of the *Gazette*, how they will re-set and experiment, patiently and tirelessly, to produce precisely the effect at which the author, not always too clearly, is

aiming. But contributors could help Messrs. MacLehose and also reduce some of the editorial drudgery by remembering that a manuscript should be written *exactly* as it is meant to appear when in print. It is not a compositor's job to expand informal abbreviations (though it is a point of honour with him to decipher even the vilest of handwritings); it is not his function to turn into decent English such slipshod and undignified phrases as

" $\therefore$ Eq. $= \odot$ ".

If a manuscript contains symbols and groups of symbols which are easy to write, but ugly and costly to transfer to type, the compositor must nevertheless reproduce them. If he finds in a block of solid text the symbols $\sqrt{ax^2+bx+c}$ or $\frac{x}{2}$ he will nobly set up these symbols with the extra cost and uglier effect which they involve, for although he knows that it is cheaper, more pleasing and equally intelligible (at least to the adult reader) to print them as $\sqrt(ax^2+bx+c)$ and $\frac{1}{2}x$ or $x/2$, he cannot do this if it is not in the script. Intending contributors would do well to study the pamphlet prepared by the London Mathematical Society, *Notes on the Preparation of Mathematical Papers*, particularly if their work involves a large proportion of symbolism. Diagrams need not be drawn in a form ready for the photographic process; it is usually more satisfactory to get the printers' draughtsman to prepare the finished drawing. But figures should be drawn on separate sheets in such a way, and with such descriptions of how to draw them, that the draughtsman, working only from the diagram and comments, not the manuscript itself, can prepare the finished drawing for reproduction. If the diagram is to be a triangle ABC with a line joining the vertex A to a point X of the base BC, it should be made quite clear, preferably by a marginal note, whether X is the midpoint of BC or the foot of the perpendicular from A, or what. During the past sixteen years I have had to re-draw about 90 per cent. of the diagrams sent to me.

To turn to general policy. Speaking broadly, the review pages are the most important, and evidently the most widely read. The reviews must therefore be ample, carefully weighed and authoritative, and to them if necessary any other part of the *Gazette* must be sacrificed. At the moment, publishing languishes, but it is hoped that we shall continue to have interesting and informative reviews in every number, and soon an increase in the number of books reviewed.

The Notes should be mainly, though not exclusively, concerned with teaching points. Here it does not matter much about originality in the strict sense, a point which readers might do well to bear in mind. Someone sends a note about some property of the conic, a proof which he has worked out and has found to go down well with his class. It is printed, and what happens? Some reader, with that glow of conscious rectitude we all feel when exposing another's shortcomings, writes in this strain: " Where can your illiterate contributor have received his so-called education, that he does not know that this proof is to be found in *Modern geometry of the conic*, by Messrs. Blank and Blankdash, published in 1897 at the Press of the University of Bad Lands, Minn., and that I myself have used this very proof in my teaching for the last 38 years." Seriously, it matters very little. As for Messrs. Blank and Blankdash, very likely they took the proof from Apollonius anyway. If it is good, and if it seems to be not widely known, then the *Gazette* is the place for it, though of course priority will be gladly acknowledged if possible. (This is not always a simple matter in elementary mathematics, and such phrases as " my proof " or " the best proof " should usually be avoided in referring to elementary work). Even a bad proof is often worth printing, as a bait to

call forth better ones. But putting it roughly, the Notes are concerned not so much with new proofs as with the spreading of knowledge of good proofs, an aim set out clearly on the cover of the first number of our journal.

One word about the Gleanings: they are not all intended to be funny, nor all serious. It might be safer, but clearly too insulting, to attach some kind of distinctive mark to those meant to be humorous, but there is really nothing to be done but to ask that each Gleaning should be taken on its own merits. Even a Note, sometimes, may be meant jokingly, though this can lead to dangerous complications.

Of criticisms of general policy that have been offered, perhaps the most frequent is that the average member does not find the whole of each issue readable and interesting, coupled generally with the corollary that there tends to be too much high-brow material and too little discussion of points in the teaching of elementary mathematics. The first complaint is flatly unreasonable: it is not to be expected that each one of 2000 readers can find the whole of every number to his taste—it is not even intended that he should. Of the numerous periodicals which an average member may be supposed to take, is there one, in the range from the *Proceedings of the London Mathematical Society* to the *Daily Moan*, of which he reads every word and of which he finds every paragraph of interest? Our aim is, and can only be, to ensure that each member of the Association will find some proportion of each number to his taste, say one article in three, one Note in two, two reviews out of three.

The suggestion that there is too much high-brow material in the *Gazette* causes me more anxiety. But some of the blame, if there be any, lies on the shoulders of members themselves, since it is not unreasonable for an editor to confess his inability to publish those contributions which he never receives. Mr. Siddons once wrote to Greenstreet and urged that more articles on the teaching of elementary mathematics should appear in the *Gazette*, and received the brief reply: "Write them." I cannot recall ever having refused an article solely on the grounds that it was too elementary; those of this type which have been rejected, on the grounds that the points raised were not of sufficiently wide interest, or had already been frequently discussed in the *Gazette*, cannot amount to more than three or four in 15 years, while very many articles have been refused because, although excellent, they appeared to be too advanced to interest more than a small proportion of our members. Moreover, it has been said from the opposite side that the *Gazette* spends too much effort on points of routine teaching and has thus thrown away its chances of being recognised as a serious mathematical periodical. As long as both comments continue to be made, there is some ground for believing that we are not too far from the golden mean.

At the first meeting of the executive committee after the outbreak of war, a discussion took place on the best way of meeting the need to reduce the size of the *Gazette*, and it was agreed that as long as books continued to be published the review space should not be cut down, and that the first economies should be effected at the expense of the more advanced articles, provided, of course, that the supply of other articles remained adequate. If the *Gazette* for 1939 is compared with that for 1944, it will be seen that in 1939 there were 37 articles, which can be classified as 8 on elementary school work, 19 on advanced school work, 10 high-brow, while the figures for 1944 give a total of 17 articles divided as 4, 10, 3. But this classification would no doubt move Mr. Gibbins' Sixth Form to scorn, that his article on three-dimensional chess should be reckoned high-brow; a sense of proportion must be preserved. It is easy to look at the title "The Laplace transformation method in circuit theory" and hastily to turn the page with the comment: "This high-brow stuff is not for me", but a little patience would reveal that the title covers a

very simple account of a highly convenient method of solving linear differential equations with constant coefficients under given initial conditions, work that is frequently part of the VIth Form course, while the illustrative examples, though taken from the domain of simple electric circuits, should not, in this radio age, be beyond the comprehension of a Science VIth. Most boys at that stage know a good deal more about circuits than we did—or do.

It should be remembered, too, that a good deal of elementary material which would once have gone into the *Gazette* now goes into our Reports.

But there is a wider and more important point of view. We are teachers of mathematics, and our teaching will be better, not worse, if we try to keep ourselves in touch with some of the developments of our subject, though these may be far beyond our normal class-room range. We cannot keep in touch with all, but we shall be better teachers if we have some idea of how some part of creative mathematics is developing. This Association is an association of teachers of mathematics ; the word " mathematics " there is as important as the word " teachers ". Let us not forget this. Recent Annual Meetings have given some impression that our duty is to teach those parts of mathematics, and only those parts, for which the physicist, in his higher wisdom, can see any use. By all means let us offer and accept cooperation, but let us be resolute against subordination. Teach mathematics first, and the applications will come the more easily ; put the applications first, accept Bacon's view of mathematics as the handmaid of natural philosophy, and we shall soon discover that it is a situation for a handmaid in which there are no evenings off and no followers allowed. This is not a matter of mere prestige, but one vital to the best interests of education ; we as an Association should insist on the importance of mathematics as a subject in the school curriculum in its own right, not as a mere utilitarian adjunct to the natural sciences. Only so will mathematics be properly appreciated and efficiently taught. And because mathematics is a central subject in its own right, so the *Gazette* must regard itself as a mathematical periodical first, and a storehouse of teaching tricks and dodges second.

It has been suggested that the interests of members would best be served by a division of the *Gazette* into two sections, marked " Elementary " and " Advanced ", a division either in each number, or by the labelling of numbers in each volume. There is the obvious difficulty of drawing a line, with the attendant risk of mortally offending some contributor's vanity by putting him in the wrong box. But there is a more serious objection to this course. Readers would soon form the habit of looking only at that section likely to suit their tastes and circumstances ; not only would they thereby miss much of interest, since who among us is bold enough to say that his interest is wholly elementary or wholly advanced, but they would soon begin to feel that they were receiving an inadequate return for their subscription, since they would regard a definite part of each number or of each volume as closed, *a priori*, to them. As things stand, there is evidence to show that many readers do not look simply at those items which from title seem likely to be of interest to them, but do now and again find something outside their usual range which appeals to them, and do derive benefit from an enlargement of interest and knowledge.

During the war, delays were a very regrettable feature of our performance, but for these there was an obvious and valid excuse. Even in time of peace, however, the author may feel that his contribution does not appear in print with sufficient rapidity, and the reader may complain that a book has ceased to be new by the time he sees a review of it in the *Gazette*. To take the author first, it must be remembered that time taken in setting up, correcting proofs and laying out a number is not to be regarded as time wasted. But even when

this has been done without any loss of time, there is still a chance of thoroughly justifiable delay. It may happen that an article must be held over because its inclusion would make the number larger than is convenient; it is better to have two numbers of 64 pages than one of 80 followed by one of 48. When this happens, the general principle is "First come, first served". But this principle itself cannot be followed blindly if adherence to it would produce an ill-balanced number—too much advanced material, too much elementary material, too much algebra, too much geometry. It is then not unfair to postpone one or two items in order to produce a better balance.

With reviews, the situation is somewhat different. Time must also be taken in finding a willing reviewer, and in giving him an opportunity to read the book, a feat some reviewers actually perform. He must then write a considered judgement, and we cannot bully our reviewers. All they get for their services is the right to retain the review copy, and while it may be that a treatise costing a couple of guineas is not to be sneezed at, experience shows that *Gazette* reviewers will take just as much time and trouble over a half-crown tract as over a more expensive and elaborate volume. The *Gazette* gets good reviewers, not in spite of the fact that they are not paid, but because the relation is one of good-will, free from all considerations of cash payments. For this reason, so Sir Edmund Whittaker was kind enough to say in a recent conversation, the *Gazette* gets the best reviews a mathematical journal in this country could hope to print. Indeed, in our Association circular, we claim that the reviews are written by men who can speak with acknowledged authority; such a claim should be frequently tested, and in fact every volume index is carefully scrutinised from this angle. For purposes of illustration here, a number was taken at random (it happened to be that for May, 1940) and in it were found the following reviewers: Professor Chapman on mathematical physics, Professor Daniell on functions of a real variable, Mr. Ferrar on algebra, Professor Massey on quantum theory, Professor Milne-Thomson on vector methods in applied mathematics, Mr. Sadler on mathematical tables, Professor Whitehead on geometry, Dr. Wishart on statistics. These mathematicians are all acknowledged experts in their respective provinces, and some of them must be regarded as the final court of appeal in this country on the matters under their review.* The care and skill which Greenstreet lavished on the *Gazette* made it easy for him to persuade the experts that the *Gazette* was a journal worthy of their help and support, and if the standard in this respect has not declined, it is due to the superb work of Greenstreet and to the kindness with which busy men give up some of their scanty leisure to the often thankless task of reviewing. But it is obvious that if they were being continually pestered to produce a review by a set date, continually being reminded that such-and-such a review was over-due, the system of good-will which has been slowly built up over so many years and has worked so well, would very soon collapse. Of course, a promise made is not always kept, and somehow when this happens it is always over a book which particularly deserves a review; but this seems to be the inevitable price we pay for a flexible and smooth-running system.

Here perhaps is the place to mention some constructive suggestions which have been made. Our staunch French friend, Professor V. Thébault, is anxious that we should devote a section to problems and solutions, questions and answers. As he says, this was a notable feature of many English periodicals

* During the war, too many reviews bore the initials T. A. A. B. But most mathematicians were overworked and unable to spare time from national duties to perform less urgent tasks. In these circumstances it was thought better to make sure of calling attention to new books, even though the reviewer could obviously make no claim to be regarded as an expert critic.

at one time; it is enough to recall the *Educational Times*, while the *Gazette* itself did have such a feature in earlier days. There is much to be said in its favour, but the matter is one for the decision of the Council; clearly a small editorial problem board would be needed, to sort, assess, select. In any event, the matter cannot be profitably discussed until we have a much greater freedom in the use of paper. Members may be reminded that, although the war is over, the size of the *Gazette* is determined not by the Association, but, effectively, by the Paper Control Department of the Ministry of Supply.

At one time, it was suggested that all addresses given to Branches of the Association should be printed without further consideration in the *Gazette*. Such a hard-and-fast rule would be, in my judgement, most undesirable. The present procedure, though less systematic, is flexible and should work well. In view of the dislocation of Branch activities by the war, the scheme may be outlined here. It is this: each Branch Secretary has been asked to send to the *Gazette* a notification of any address which he may consider suitable for publication. This leaves scope for editorial consideration of the material on its own merits, and in conjunction with other articles recently printed, or about to be printed, so that overlapping can be avoided.* Before the war, Branch Secretaries were perhaps too critical, and did not avail themselves as frequently as they might have done of their opportunities. At present, this is not to be deplored, and it would be desirable for them to remain hypercritical till we have more freedom in the use of paper.

The President (Professor Chapman) has made a particularly felicitous suggestion, which will shortly be put into practice. It is that each number should have, as a frontispiece, a picture of some mathematical worthy or of the title-page of some rare or noteworthy mathematical treatise. With this there would go some paragraphs of explanatory biographical or bibliographical text. Members can help in this admirable scheme by sending suggestions and offering suitable pictures for reproduction.†

Finally, I hope I have not seemed too complacent, that I have not given the impression that I consider the *Gazette* perfect in organisation and open to no kind of improvement, for all editors should humbly bear in mind Hazlitt's comment on Gifford, the editor of the *Quarterly Review*: "He is admirably qualified for this situation by a happy combination of defects, natural and acquired." The *Gazette* can and shall be improved, but only if all members will cooperate. Let us increase our membership by 50 per cent., or, better still, let us double it. The effect of the increase would be translated at once into a bigger and better *Gazette*. For us to work to this end would be in the spirit of that nobler sentence of Bacon's, which at Greenstreet's happy suggestion was taken as the motto of our Association: "I hold every man a debtor to his profession, from the which as men do of course seek to receive countenance and profit, so ought they of duty to endeavour themselves by way of amends to be a help and an ornament thereunto."

T. A. A. B.

* The revived machinery of the Branches Sub-committee will slightly affect the details but not the principle of this scheme.

† The first of the series appears in this number of the *Gazette*, a photograph of the title-page of the first printed Euclid in Greek, Grynaeus' edition of 1533.

All suggestions will be welcomed and carefully considered, especially if an indication of means for obtaining a photograph is added. But photographs should not be sent until it is certain that they can be used, and members are advised that in any event it is undesirable to trust negatives on glass to the post. Biographical or bibliographical details suitable for a paragraph of comment will also be welcomed.

WHY DOES A BICYCLE KEEP UPRIGHT?

By F. G. Maunsell.

1. *General Considerations.*

Most of us have asked this question at some time or another; generally we have received some such answer as: "It's a gyroscopic effect, of course". Further details are not easy to obtain, and the majority of books on Mechanics do not mention the subject.

I have, however, found some references, which seem to be of sufficient interest to be quoted. They are as follows:

(i) *The "Encyclopedia Britannica"*. The thirteenth edition has nothing of import in the articles on "Bicycle" and "Cycling": the fourteenth edition (under "Bicycle") has the following sentence, which does not, indeed, purport to answer the question of uprightness, but which is, perhaps, of some interest: "The slope or steering angle is the cause of a bicycle running naturally in a straight line, because any movement of the wheel out of the straight tends to raise the bicycle slightly: therefore the weight of the bicycle and rider discourages any such deflection of the front wheel . . .". I may add that nothing before or after helps to elucidate this remarkable theory.

(ii) *A. Gray*: "*A Treatise on Gyrostatics and Rotational Motion*", 1918. In Chap. VII, para. 12, appears the following explanation: "If the rider feels himself beginning to fall over to one side or the other he instinctively turns the bicycle towards that side, and the inertia in the forward movement, assisted by the gyrostatic action of the driving wheel, over which the rider sits, causes the bicycle frame to set itself erect again".

(iii) *R. W. Pohl*: "*Physical Principles of Mechanics and Acoustics*", *English translation*, 1932. Chap. VII, para. 11, gives the following explanation, headed "Riding a bicycle without touching the handlebars": "Suppose the rider tilts a little towards the right . . . the front wheel, acting as a gyrostat, describes a precessional motion about the vertical and curves to the right. The line joining the points of contact of the front and back wheels with the ground is again brought under the centre of gravity of the rider, that is, the point of support is brought back under the centre of gravity."

(iv) *Appell*: *Mécanique Rationnelle*, II, has a section on bicycles (para. 422). He apparently holds the view that the cyclist deliberately turns the handlebars towards the side to which he is falling. This turning of the handlebars brings the bicycle upright again by means of centrifugal action. Appell gives references to four papers on the bicycle and to a book by Bourlet—*Nouveau traité des bicycles et bicyclettes*. I have not been able to consult this book, but it is referred to in Bourlet's paper mentioned below and apparently contains no more new theories.

The four papers are as follows:

(v) Bourlet: *Bull. Soc. Math.*, 1899.

(vi) Boussinesq: *Jordan's Journal de Math.*, 1899.

(vii) Routh: *Mess. of Math.*, 1898–9.

(viii) Carvallo: *Journ. de l'École Poly.* (V, VI, 1900).

All four writers take, broadly speaking, the same views as Appell. Carvallo alone goes somewhat further and considers the case of riding without touching the handlebars. He alone appears to appreciate the construction of the front fork which, as I shall explain later, ensures that turning the handlebars lowers the centre of gravity. Unfortunately (from my point of view, at any rate) Carvallo treats the problem entirely analytically, and I have not yet had time to follow out in full the long and involved calculations of his paper (which runs to more than 100 pages). I hope to do so in the near future.

He reaches the remarkable result that a bicycle ridden without touching the handlebars is stable for a band of velocities. That is, it is unstable for velocities both very low and very high. However, the mechanism of this result is not clear.

(ix) The Editor of *Cycling* informs me that the matter was put before the Brains Trust in December 1942, and their reply was printed in *Cycling* in their issue of January 6, 1943. Unfortunately, he could not supply me with a copy of this article, and I should be very much obliged if any member of the Mathematical Association who happens to possess a copy could lend it to me for a short period.

(x) The Editor of *Cycling* also sent me the views of his technical expert. The views he puts forward are that the principal cause of a bicycle remaining upright is the deliberate balancing by the rider moving the handlebars from side to side, thus moving the point of support under the centre of gravity of the machine and rider. He also states that a fixed-wheel machine can be balanced at very slow speeds and even when stationary, although a free-wheel one cannot. Accepting this as an experimental fact, I confess I cannot see the reason for it.

Even these few quotations show that there exists a certain confusion of ideas on the subject, and I will now attempt to classify the theories that I have heard put forward to explain the effect. All but one of these connect (correctly, I believe) the restoration of uprightness with the turning of the handlebars towards the side to which the bicycle is falling. The exception is:

Theory A. This asserts simply that a tendency to fall to the right is counteracted by the rider shifting his weight to the left. This action is certainly not a conscious one, and is presumably a subconscious reaction acquired when learning to ride. The theory is, however, certainly incorrect. If it were true, it would be as easy to balance a bicycle in which the handlebars were rigidly prevented from turning or a stationary bicycle as a bicycle ridden in the ordinary way. On the contrary, these two feats are extremely difficult to accomplish, if not impossible.

All the other theories agree that when the bicycle starts to fall to the right, the handlebars are turned to the right, and that this action causes the bicycle to regain the upright position. There are therefore two questions to be answered. Firstly, "Why do the handlebars turn to the right when the bicycle starts to fall to the right?" Secondly, "Why does turning the handlebars to the right when the bicycle starts to fall to the right have the effect of causing the bicycle to regain the upright position?"

The theories put forward to account for the first effect are as follows:

*Theory B*1. This is the theory, as put forward by Gray in the second of the above quotations, that the turn is due to the instinctive (subconscious is perhaps a better word) action of the rider. The simplest answer to this is the fact that it is possible to ride a bicycle without holding the handlebars.

*Theory B*2. The theory, given by Pohl, that the turning of the handlebars is due to the gyrostatic precession of the front wheel. That this effect exists is undoubted. Later in this paper I hope to show that it is of secondary importance compared to a further effect.

*Theory B*3. This further effect I have called the "statical" effect. It depends on the fact that when the bicycle is tilted slightly to the right, a slight turn of the handlebars to the right lowers the centre of gravity, while a slight turn to the left raises it. The effect is present whether the bicycle is moving or stationary and can be tested immediately by anyone with his own bicycle in the latter case.

So much for why the handlebars are turned. Now for theories why this turning causes the bicycle to come upright again.

*Theory C*1. Gray's theory that it is due to gyrostatic action on the back wheel (" driving wheel " presumably means this). Some such action there will be : I shall show that it is much too small quantitatively.

*Theory C*2. Pohl's theory that the turning of the handlebars shifts the line joining the points of contact of the front and back wheels with the ground sounds plausible. Unfortunately, it is in direct conflict with the facts. The line of the front head, *i.e.* the axis about which the front wheel turns when the handlebars are turned, meets the ground about three inches in front of the point of contact of the front wheel when the bicycle is upright. Consequently, for small deflections of the front wheel to the right, the point of contact of the front wheel goes to the left, and the bicycle is, due to this cause, more and not less inclined to fall to the right. It is true that a *large* deflection to the right will send the point of contact to the right, due to the forward bend in the front forks. This is why, if you try to balance on a stationary bicycle, a tendency to a fall to the right can be stopped by a violent right-hand turn to the handlebars. We are, however, concerned only with *small* deflections in the problem of the moving bicycle.

*Theory C*3. The completely satisfactory explanation of this part of the effect is that it is due to " centrifugal force ". In other words, turning the handlebars to the right makes the bicycle describe a curve to the right. Thus the bicycle is given an acceleration normal to the curve, that is, to the right. The only forces that can produce this acceleration are sideways reactions on the points of contact of the wheels with the ground, and these reactions, directed to the right, have a moment about the centre of gravity which forces the machine to rise again to the upright position.

To sum up, the explanation I put forward as the correct one is compounded of theories *B*3 and *C*3. When the bicycle starts to fall to the right, the " statical " effect causes the handlebars to turn to the right and the " centrifugal " effect causes the bicycle to return to the upright position. The function of the rider's hands is simply to act as a steadying agent and prevent a too violent oscillation occurring.

It is agreed that the gyrostatic effects *B*2 and *C*1 tend to act in the same way. In order to decide between *B*2 and *B*3 and between *C*1 and *C*3 a quantitative investigation is necessary.

2. *Quantitative Work.*

As we only require to get an idea of the relative orders of magnitude of the effects, quite rough estimates of the various weights and lengths involved will be sufficient for our purpose. In what follows I have obtained some of the data by rough measurements of my own bicycle, others by estimation.

m = mass of front wheel = 4·25 lb.
r = radius of front wheel = 1·1 ft.
k = radius of gyration of front wheel about its axle = 0·4 ft.
M = mass of bicycle plus rider = 175 lb.
a = height of c.g. of bicycle plus rider = 3·5 ft.
b = distance between back and front axles = 3·5 ft.
λ = ratio of weight on back to weight on front wheel = 4.
α = rake of front head from vertical = 30°.
d = distance in front of point of contact with ground that prolongation of front head meets ground = 3 in. = 0·25 ft.
v = velocity of bicycle, say, 10 ft./sec.
ψ = sideways tilt of the bicycle.

From the above are derived :

n = angular velocity of front wheel = 9·1 radians/sec.

C = moment of inertia of front wheel about its axle = 0·7 lb.-ft.^{2}

The gyroscopic effect B2.

When the bicycle starts to fall sideways, we require the angular velocity $\dot{\psi}$ about a horizontal axis. This, of course, increases from zero as ψ increases from zero. Taking the mass as concentrated at the C.G., by the ordinary formula :

$$\tfrac{1}{2} \cdot 175 \cdot 3{\cdot}5^2\dot{\psi}^2 = 175 \cdot 32 \cdot 3{\cdot}5\,(1 - \cos\psi).$$

This gives for $\psi = 1°$, $\dot{\psi} = 0{\cdot}05$ radians/sec.

$\psi = 5°$, $\dot{\psi} = 0{\cdot}26$ radians/sec.

Now the couple L_1 acting on the front wheel and tending to turn it to the right $= Cn\dot{\psi} = 0{\cdot}7 \times 9{\cdot}1\dot{\psi}$.

Hence for $\psi = 1°$, $L_1 = 0{\cdot}3$ pdls.-ft.

$\psi = 5°$, $L_1 = 1{\cdot}7$ pdls.-ft.

This couple L_1 is about a vertical line : the resolved part about the axis formed by the front head is $L_1 \cos 30°$, or 0·25, 1·5 pdls.-ft. respectively.

The statical effect B3.

We suppose the bicycle has tilted an angle ψ to the right, and we require the couple due to the weight which tends to turn the front wheel to the right. To find this we suppose the handlebars turned a small angle β, and we work out how far the centre of gravity sinks.

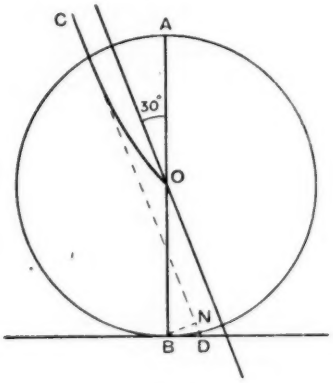


FIG. 1.

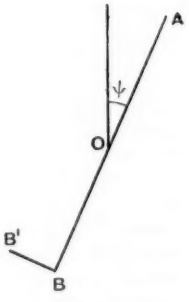


FIG. 2.

Fig. 1 represents the side elevation of the front wheel when the bicycle is upright. AOB is the vertical diameter. CND represents the line of the front head. BN is perpendicular to CD and $BN = BD \cos 30° = 0{\cdot}25 \cos 30° = 0{\cdot}21$ ft.

Fig. 2 represents the rear elevation of the front wheel when the bicycle is tilted at an angle ψ to the right.

When the front wheel is turned an angle β to the right, B describes (to the first order) a circle about N. So B moves to the neighbouring point B' where $BB' = BN \cdot \alpha = 0{\cdot}21\alpha$ ft.

Hence O sinks a distance $BB' \sin \psi = 0{\cdot}21\alpha \sin \psi$.

The C.G. of the above machine will therefore sink one-fifth of this distance, that is, $0{\cdot}04\ \alpha \sin \psi$.

Hence the work done by gravity $= 175 \cdot 32 \cdot 0{\cdot}04\ \alpha \sin \psi$ ft.-pdls.

The work done against the couple L_2 which is required to keep the front wheel from turning to the right is $L_2\alpha$, and this must be equal to the former expression.

That is, $L_2 = 175 \cdot 32 \cdot 0{\cdot}04 \sin \psi$.

If $\psi = 1°$, $L_2 = 4$ pdls.-ft.

$\psi = 5°$, $L_2 = 20$ pdls.-ft.

It will be seen that the " statical " effect $C3$ is nearly 10 times as great as the gyroscopic effect $C2$ even for the comparatively large tilt of 5°, and is proportionately more for smaller tilts. It is undoubtedly the major factor in causing the handlebars to turn in the direction of tilt.

The gyroscopic effect C1.

Suppose the handlebars are turned an angle ψ. The bicycle will describe a circle to right whose radius ρ is given by the formula $b/\rho = \psi$ (where ψ is in radians).

Thus when $\psi = 1°$, $\rho = \dfrac{3{\cdot}5 \times 180}{\pi}$ ft. $= 200$ ft. ;

$\psi = 5°$, $\rho = 40$ ft.

Since we assume the velocity to be 10 ft./sec., the angular velocity $\dot{\psi}$ with which this circle is described is

$\frac{1}{20}$, $\frac{1}{4}$ rad./sec.

in the two cases.

The couple L_3, which tends to right the bicycle due to the precessional action on one wheel, is $Cn\dot{\psi}$.

The value of the couple L_3 is, as in $B2$,

for $\psi = 1°$, $L_3 = 0{\cdot}3$ pdls./ft.

$\psi = 5°$, $L_3 = 1{\cdot}5$ pdls./ft.

If we allow that *both* wheels give a precessional effect, we get

for $\psi = 1°$, $L_3 = 0{\cdot}6$ pdls./ft.

$\psi = 5°$, $L_3 = 3$ pdls./ft.

The " centrifugal " effect C3.

As before, we consider the handlebars turned angles 1° or 5°.

The normal acceleration towards the right is then $\dot{\psi}^2\rho$, or $\frac{200}{400}$, $\frac{40}{16}$ respectively,

i.e. $\frac{1}{2}$ ft./sec.^{2}, $\frac{5}{2}$ ft./sec.^{2} ;

the righting couple is $M \times$ acceleration $\times$ height of centre of gravity, or

$$175 \times 3{\cdot}5 \times \text{acceleration}.$$

This works out at 300, 1500 pdls.-ft. respectively, and it is clear that the effect $C1$ is insignificant compared to the effect $C3$.

F. G. MAUNSELL.

SIGN, AND ELEMENTARY VECTOR IDEAS, IN PLANE ANALYTICAL GEOMETRY AND TRIGONOMETRY.

By D. K. Picken.

1. Plane Analytical Geometry, and elementary Trigonometry, are the beginnings of the systematic application of mathematical analysis to Geometry, *i.e.* of systematic expression in terms of "number"—which is *essentially* "*real*" *number*.*

Complete application of the *generality* of the algebra (within the restriction to *real* number) is, of course, the power of the method; and, in particular, *sign of the (real) numbers is significant throughout.*

2. *Sign by ratio.*

The fundamental connecting-link between Geometry and Analysis is the relation between "Real Number" and "Ratio", expressed by

$$x = \overrightarrow{OP} : \overrightarrow{OA}, \quad \text{or} \quad \overrightarrow{OP} = x.\overrightarrow{OA},$$

where P is the variable point collinear with given points O, A †; *the number*, x, *being positive or negative according as the two vectors* OA, OP *have the same directions or have opposite directions.*‡

3.1. This basic principle—with obvious corollaries, in terms of equivalent specifications of a given (non-localised) vector—is adequate to all relevant problems of sign, in the field in question. Thus: §

3.2.1. *Cartesian coordinates.*

$$x = OM : OA, \quad y = ON : OB,$$

where A, B are the "unit-points", $(+1, 0)$, $(0, +1)$, on the (directed) axial lines, and M, N are the points of these lines respectively, such that the lines PN, PM are the respective parallels through P. (OM, ON are "the components of OP in the given directions Ox, Oy". *Vide* § 6.1 (5), *infra.*)

3.2.2. *Trigonometric (or circular) functions.*

$$\cos AOP = OM : OA, \quad \sin AOP = ON : OB,$$

if A, B, P are on a circle (centre O) and $\angle AOB$ is a *positive* right angle; $\tan = \sin/\cos$, etc. ||

* It is necessary to insist that the (so-called) "unreal" aspects of the algebra are *not* applicable to the *plane* geometry. They are, of course, applicable to the four-dimensional geometry which *fully* corresponds to the algebra—and *includes* the plane geometry ("section" fashion: see the second part of Note 1317, XXII, No. 251, p. 394).

† Vectors of this type are the main theme of this article; in what follows, the arrowhead vector sign is to be "understood", throughout.

‡ Opinions vary as to the quite fundamental importance of this proposition; but the *fact*, as stated, is common ground. The present writer has made plain his own position, in relevant communications; *vide*, in particular, XXII, No. 250, pp. 225–233.

§ Some of what immediately follows was set out more fully in XI, No. 165, pp. 330–1 (July, 1923). A minimum of repetition is unavoidable. (I have in my copy of that number a letter from E. H. N. on the subject—which was the beginning of a personal contact I have greatly valued.)

|| The trigonometric functions "cos" and "sin" are, thus, a specially important particular case of *Rectangular* Cartesian coordinates.

3.3. *Polar coordinates.*

Using the same " initial line " of reference, θ is the (measure of the) *general* inclination of the *line* OP to the *direction* of reference OA : it is a function of position with a double infinity of simply-related values. And, if U be the " unit-point " in the θ-direction,

$$r = OP : OU ;$$

it is a *two*-valued function, the sum of the two values being zero (and each corresponding to a single infinity of values of θ).

These coordinates are related to *Rectangular* Cartesian coordinates (referring to O, A as above) by

$$x = r.\cos\theta, \quad y = r.\sin\theta,$$

which are quite *general* (in terms of the many-valuedness of θ, r). *Vide* § 6.1(4), *infra*.

There is much to be said for writing the Polar coordinates in the order θ, r (rather than r, θ).

3.4. *Parametric specifications of straight line.*

3.4.1. Given point, P_1, and direction—by ψ-angle for the rectangular frame, by direction-coordinates l, m for the general Cartesian frame—we get the *general* forms :

$$x = x_1 + \rho.\cos\psi, \quad y = y_1 + \rho.\sin\psi ;$$
$$x = x_1 + l.\rho, \qquad y = y_1 + m.\rho,$$

where $\rho = P_1P : P_1V$ and V is at unit distance from P_1 in the given ψ-direction.*

3.4.2. Given two points, P_1, P_2 : if λ is the " position-ratio " $P_1P : PP_2$, then

$$\lambda = (x - x_1)/(x_2 - x) = (y - y_1)/(y_2 - y) ;$$

hence $$x = (x_1 + \lambda.x_2)/(1 + \lambda), \quad y = (y_1 + \lambda.y_2)/(1 + \lambda),$$

are *general* forms.†

4. " *Sign of the perpendicular.*"

Line specified by the general first degree equation

$$(L =)\ a.x + b.y + c = 0,$$ ‡

with reference to a *Rectangular Frame.*§

* In " equation " form, $(x - x_1)/l = (y - y_1)/m\ (= \rho)$; also, $y = \mu\,.\,x + \beta$, or $x = \nu\,.\,y + \alpha$ —where $\mu\ (= m/l)$ is the gradient, $\nu\ (= 1/\mu)$ the co-gradient, of the line, and $(\alpha, 0)$, $(0, \beta)$ its axial points. (The " Change-of-origin " transformation of coordinates is, of course, used ; *vide* § 6.2.1, *infra*.)

† In " equation " form,

$$(y_1 - y_2).x - (x_1 - x_2).y + (x_1.y_2 - x_2.y_1) = 0 ;$$

also, $x/\alpha + y/\beta = 1$, with notation as above.

‡ Comparing with the footnotes to § 3.4, *supra*—

$$a.l + b.m = 0 ; \quad \mu = -a/b, \quad \nu = -b/a ; \quad \alpha = -c/a, \quad \beta = -c/b$$

—in relation to which the vanishing of one, or of two, of the coefficients a, b, c is properly interpretable.

§ Questions of perpendicularity, or of distance—or of the use of trigonometric methods—will obviously be simpler when the frame of reference is *rectangular*. But it is a profound mistake to introduce Analytical Geometry, generally, in terms of the Rectangular Frame. Most of the elementary theory can—and should—be expressed in terms of the general Cartesian Frame (which should not be stigmatized by the term " oblique "!). This point becomes of first-rate importance in the treatment of 3-dimensional geometry. (See next footnote.)

4.1.1. *The "perpendicular form" of the equation.*

This standard method is the best and the most fundamental.* For it, we use specification of the line by means of the point (K), which is the "foot of the perpendicular" from O—and specification of that point by polar co-ordinates, say, $\theta = \chi$, $r = p$, as defined in § 3.3, *supra*. Then, either by § 3.4.1, or, more directly, by orthogonal projection, the equation takes the form

$$\cos\chi.x + \sin\chi.y = p.$$

Transformation to the "Parallel Frame" (§ 6.2.1, *infra*) with origin P_1 gives the ξ, η equation: $\cos\chi.\xi + \sin\chi.\eta = p - \cos\chi.x_1 - \sin\chi.y_1$ ($= \varpi$, say); and ϖ is, therefore, the measure of the perpendicular from P_1 to the given line—its sign being interpretable from the fact that χ, ϖ are polar coordinates, with reference to $P_1\xi$, of the "foot of the perpendicular" from P_1.

4.1.2. *General expression for the perpendicular.*

The relations $\quad (\cos\chi)/a = (\sin\chi)/b\,(= 1/k) = p/(-c)$

give $\varpi = -L_1/k$, where $k = \sqrt{(a^2+b^2)}$—choice of sign for k † being choice of the χ-direction, by reference to which the sign of ϖ is interpreted. And the "foot of the perpendicular" is the point

$$(x_1 - aL_1/k^2,\ y_1 - bL_1/k^2),$$

where $k^2 = a^2 + b^2$ and is *unambiguous*.

4.2. *Alternative methods.*

4.2.1. By direct application of § 3.4.1, giving

$$a(x_1 + \varpi\,.\cos\chi) + b(y_1 + \varpi\,.\sin\chi) + c = 0,$$

where $\chi = \psi + \frac{1}{2}\pi$ and $(\cos\psi)/b = (\sin\psi)/(-a) = 1/k$.‡

* This is not quite obvious, in the case of 2-dimensional geometry—because this case (of $n = 2$) is too special and too simple, and the methods used in it not, therefore, sufficiently "systematic". What this means becomes clear when we compare with the 3-dimensional case—where the methods used depend much less specially on the particular value of $n\,(=3)$. Some parallel treatment—however elementary—of the cases of $n = 2$ and $n = 3$ is of great value, for sound grasp of the theory. In the latter case, the "Perpendicular Form" of the equation to the Plane is, of course, quite fundamental—for the *general* Cartesian frame. (It is to be specially noted that the mistaken emphasis on the *Rectangular* Frame (see previous footnote) has led to over-emphasis upon direction-*cosines*—where these are, however important, properly subordinate to the so-called "direction-*ratios*", which are better thought of as direction-*coordinates*. Very commonly the direction-cosines, in the case of a Rectangular Frame, are really "direction-ratios": *e.g.* § 3.4.1, *supra*. Here, of course, they are not.)

† The choice is arbitrary—but is "followed through". As between *positive* and *negative*, there is (*caeteris paribus*) general priority of the former. It is the correlation of χ and p (and ϖ)—*per* k—that matters. *E.g.* for $x - \sqrt{3}.y - 4 = 0$, $\chi = -\pi/3, \frac{5}{3}\pi$, etc., $p = 2$ or $\chi = 2\pi/3,\ -4\pi/3$, etc., $p = -2$.

‡ This is a good method for the case of the *rectangular* frame; but not nearly so good for the general frame—whereas the method of § 4.1 is not much less simple for the general frame: the "Perpendicular Form" equation being then

$$\cos\alpha.x + \cos\beta.y = p, \text{ with } \alpha = O + \beta,$$

in terms of "direction-*cosines*" for the perpendicular. For § 4.1.2 we have then the relations

$$(\cos\alpha)/a = (\cos\beta)/b\ \ (= \sin O/k) = p/(-c);$$

and $\varpi = (-L_1\,.\sin O)/k$, where $k = \sqrt{(a^2+b^2-2.a.b.\cos O)}$; etc. (It is to be noted that the symbols α, β are here used with a different—standard—meaning from that of the footnotes to § 3.4 and the first footnote to § 4. Practically all attempts at systematic notation, in terms of alphabetical symbols, almost inevitably break down; but we should, nevertheless, do the best we can.)

4.2.2. By intersection of the given line with the perpendicular line through P_1 (but this leads to the expression for ϖ^2, and does not naturally resolve the problem of sign of ϖ).

5. "*The line* (a, b, c)."

5.1. Using this notation for the line specified by the equation

$$(L=)\,a.x+b.y+c=0,*$$

two such specifications belong to one and the same straight line if, and only if, the proportion $a : b : c$ is the same for both ; but, with this fact clearly in mind, it is important to observe the distinction between one set of (real) values of a, b, c and another—whether they belong to different lines or not.† In fact, we always do use a definite (a, b, c), and should not confuse it with the other specifications to which it is geometrically equivalent ; ‡ nor is it sound to impose any such arbitrary restriction as that a, or b, or c shall be positive (though, *caeteris paribus*, a is commonly chosen positive).

5.2. From

$$L=a.(x-\nu.y-\alpha)=b.(y-\mu.x-\beta)=c.(1-x/\alpha-y/\beta),\S$$

it is apparent that L (which $=0$ for points on the line) is positive for points to one side of it,‖ and negative for points to the other side—being (more specifically)

(1) of the same sign as a for points to the same side as $(x, 0)$ when $x\to+\infty$;
(2) of the same sign as b for points to the same side as $(0, y)$ when $y\to+\infty$;
(3) of the same sign as c for points to the same side as the origin O ; ¶

and it is quite a satisfactory method (when convenient, as in § 5.3 *infra*) to distinguish, as "positive" and "negative", respectively, the two "sides" —or *aspects*—of "the line (a, b, c)" according as L is positive or negative for the points of the region in question : not (as Note 1595 rightly insists) an absolute geometrical distinction, but relative to the "scaffolding" of "the frame of reference" and to the given specification (a, b, c).

* Phrases like "the line $a.x+b.y+c=0$", or "the line L", are typical of a sketchy unprecision which is unworthy of the most exact of all branches of learning. This kind of unprecision has had very serious consequences in the Physical Sciences ; and these consequences are becoming of critical importance, in the utter inadequacy of many forms of loose expression to the extreme difficulty and subtlety of modern physical theory.

† The notation (a, b, c) for lines is, of course, analogous to the standard notation (x, y) for points. Complete analogy can be achieved in one or other of two ways : (1) by using equations of some such form as $a.x+b.y\pm1=0$ and calling the line "(a, b)"—then a *unique* specification ; this is sometimes useful ; (2) by using *homogeneous* forms, and corresponding specification of points (in the plane) as "(x, y, z)" ; this is related to matters of major theoretical importance upon which it is not possible to elaborate here.

‡ Though we may, of course, make use of these equivalents as freely as we please. (*Vide*, *e.g.*, the numerical detail in § 5.3.2, *infra*.)

§ *Vide* footnotes to § 3.4, for the notation.

‖ "To", perhaps, rather than *on*—which is commonly used for the points which constitute the line.

¶ These are, of course, consistent with one another. Taken together, they cover the special cases of the vanishing of one, or of two, of the three coefficients. It should be observed, as characteristic of the "analytical" theory, that the apparent unsymmetry of the three modes of discrimination is resolved when Cartesian Frame is realised as "limiting case" of Triangle of Reference. (The cases of § 5.3 *infra*—as shown in the figure—may be taken as examples.)

5.3. *Discrimination of bisectors of Complete Angle.*

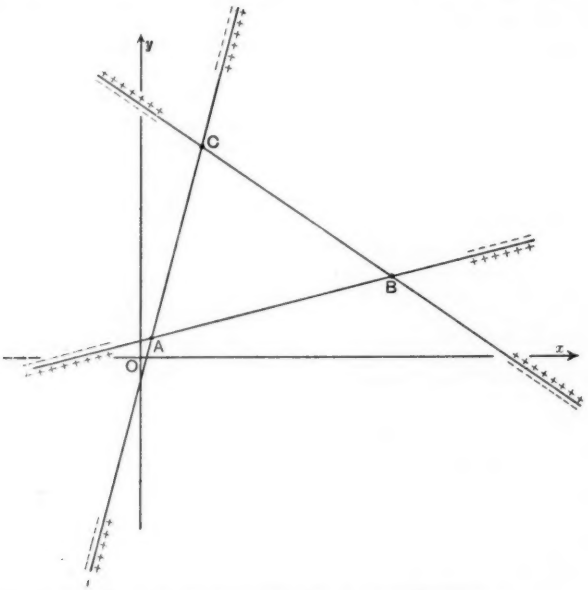
5.3.1. Given (a_1, b_1, c_1) and (a_2, b_2, c_2), the lines

$$(a_1/k_1 \mp a_2/k_2, \quad b_1/k_1 \mp b_2/k_2, \quad c_1/k_1 \mp c_2/k_2),$$

are the bisectors which lie respectively in the $(\pm, \pm)$ and the $(\pm, \mp)$ regions, as defined in § 5.2 (the frame being rectangular).

5.3.2. *Illustration.*

Applying to the "important example" used in Note 1594 (with acknowledgments), the given points A, B, C are (1, 2), (25, 8), (9, 21), and the lines BC, CA, AB are specifiable as (13, 16, −453), (19, −8, −3), (1, −4, 7)—their "aspects" (relative to that specification) being as shown in the figure.



The respective *internal* bisectors are, thence, specifiable as

$$(19+5.1, \ldots), \quad (5.1-13, \ldots), \quad (13+19, \ldots),$$

and thence, again, as

$$(6, -7, 8), \quad (2, 9, -122), \quad (4, 1, -57);$$

and the respective *external* bisectors as

$$(19-5.1, \ldots), \quad (5.1+13, \ldots), \quad (13-19, \ldots);$$

or, again,

$$(7, 6, -19), \quad (9, -2, -209), \quad (3, -12, 225);$$

and I, I_1, I_2, I_3 are obtainable from the proper combinations of these six lines.

The figure has been drawn* with the minimum of detail necessary to make all the essential facts evident (to the imagination, at least!).

6. *Transformation of Cartesian coordinates.*

Change of Frame of Reference from xy-frame to $\xi\eta$-frame—where the latter is specified, relative to the former, by origin Ω $(x=h, y=k)$ and inclination (ϕ-angle) of $\Omega\xi$ to Ox; the angles of the respective frames (or the circular measures thereof) being denoted by o (for inclination of OY to OX) and ω ($\Omega\eta$ to $\Omega\xi$).

6.1. *Vector methods.*

It is simplest to use elementary Vector ideas (including that way of expressing the method of Orthogonal Projection); thus:

(1) *Components of a vector are any set of vectors of which it is the "sum";* but, more particularly:

(2) "*The components of a given vector in two given directions*" (which are *coplanar* with its direction) *are the positive or negative vectors—with those directions, respectively—of which it is the sum.*

(Similarly—and more generally—"the components of a given vector in *any three* given, non-coplanar, directions.")

The importance of this specification is, of course, that there is (in each of these cases) *one, and only one, such set of components*: a "component in a given direction" being more strictly, a vector which has *either that direction or its opposite.*

(3) If the "two given directions" are at right angles to one another, the components are called *rectangular components*; and

the "resolute of a given vector in a given direction" is the positive or negative vector, with that direction, which is one of a pair of rectangular components of the given vector.

(4) A positive or negative vector, OP, "with a given direction" is, of course, specifiable by a product $r.OU$, where OU is the unit (positive) vector with the given direction, and r is a (positive or negative) real number. And it is obvious that

components of OP are, respectively, the "multiples" by r of the corresponding components of OU; and, in particular,

$$\textit{resolute of } OP = r.(\textit{resolute of } OU)$$

—for any given direction of resolution;

also that

the "resolute of OP in any given direction" is equal to the sum of the "resolutes, in that direction, of any set of components of OP"; etc., etc.

These are propositions of fundamental importance.

(5) The Cartesian coordinates x, y of the point P—as defined in § 3.2.1, *supra*—are the "real" numbers such that $x.OA$, $y.OB$ are "the components of OP in the given directions Ox, Oy" (and, similarly, x, y, z, for the geometry of Space).

(6) So, again, the trigonometric functions of $\angle AOP$—as defined in § 3.2.2, *supra*. (A case of *rectangular* components and, therefore, of *resolutes*.)

6.2.1. *Change-of-origin transformation.*

This is the case in which $\Omega\xi$, $\Omega\eta$ are, respectively, *the same directions* as

* Drawn for me—as on previous occasions where skill was required—by Mr. J. M. Allen, M.A., B.Sc., of Melbourne Technical College.

Ox, Oy (not merely " parallel "). Using an obvious notation, the relation

$$OP = O\Omega + \Omega P$$

gives

$$OM = OH + \Omega\mu = h.OA + \xi.\Omega\alpha = (h + \xi).OA\ ;$$

hence $x = h + \xi$, and, similarly, $y = k + \eta$.*

6.2.2. *The general homogeneous Cartesian transformation.*

This is the general case in which there is *no change of origin* (Ω at O).

Using the pair of rectangular components of OP, based on the direction Ox, *i.e.* the " resolute in that direction " and the " complementary resolute ", and with notation $R(OP)$ for *resolute of the vector* OP, etc. :

$$R(OP) = R(OM) + R(ON) = R(O\mu) + R(O\nu)$$
$$= x.R(OA) + y.R(OB) = \xi.R(O\alpha) + \eta.R(O\beta),$$

the respective directions of resolution give

$$x.OA + y.\cos o.OA = \xi.\cos\phi.OA + \eta.\cos(\phi + \omega)\dagger.OA,$$

hence $$x + y.\cos o = \xi.\cos\phi + \eta.\cos(\phi + \omega)\ ;$$

and, similarly $$y.\sin o = \xi.\sin\phi + \eta.\sin(\phi + \omega).$$

These *general* relations are " simultaneous equations " which can be " solved " for x, y in terms of ξ, η—or for ξ, η in terms of x, y.

[But the results so obtained for x, or for ξ and for η, can be written down—by analogy with that for y—by using " the *complementary* resolutes " for the cases of Oy, and $O\eta$ and $O\xi$, respectively ; *e.g.*

$$x.\sin(-o) = \xi.\sin(-o + \phi) + \eta.\sin(-o + \phi + \omega),\dagger$$

or $$x.\sin o = \xi.\sin(o - \phi) + \eta.\sin(o - \phi - \omega).$$

Or, yet again, the forms for ξ, η, respectively, can be written down from those for x, y ; *e.g.*

$$\eta.\sin\omega = x.\sin(-\phi) + y.\sin(-\phi + o)$$
$$= -x.\sin\phi + y.\sin(o - \phi).]$$

The great simplicity of the general method—when applied quite systematically, *in terms of sign of vectors* (" *in standard directions* ") *and of the specifying angles*—is to be noted. (The emphasis is on the *method*, rather than on the actual results. *Vide infra.*)

6.3. *Standard particular cases.*

Two special particular cases are important. They are best dealt with by *applying directly* the method of § 6.2.2.

6.3.1.1. *Homogeneous transformation to rectangular coordinates.*

Here the $\xi\eta$-frame is Rectangular, and $O\xi$ is coincident with Ox. Thus

$$\xi = x + y.\cos o, \quad \eta = y.\sin o\ ;$$

and, thence, $$x = \xi - \eta.\cot o, \quad y = \eta.\operatorname{cosec} o.$$

* From the point of view of the *general method* used, note the importance of writing $h + \xi$ as contrasted with (the algebraically equivalent) $\xi + h$, etc., in § 6.2.1; and, similarly, $\phi + \omega$, etc., in § 6.2.2 ; and so on. (The method is, of course, used in good textbooks, but is not sufficiently " followed through " in the detail which is so peculiarly important to this kind of discussion.)

† See the preceding footnote.

6.3.1.2. *Transformation to polar coordinates.*

If θ, r are polar coordinates of P, with reference to Ox, then (using § **3.3**, *supra*)

$$r.\cos\theta(=\xi)=x+y.\cos o,\ r.\sin\theta(=\eta)=y.\sin o\ ;$$

whence
$$r^2=x^2+y^2+2.x.y.\cos o,$$

and
$$\tan\theta=y.\sin o/(x+y.\cos o)=m.\sin o/(l+m.\cos o)$$

—if l, m are the " direction-ratios " of the line OP (either direction, here).

6.3.2. *Homogeneous rectangular Cartesian transformation.*

Here both frames are Rectangular ($o=\omega=+\pi/2$).* Thus

$$x=\xi.\cos\phi+\eta.\cos(\phi+\pi/2)=\xi.\cos\phi-\eta.\sin\phi,$$
$$y=\xi.\sin\phi+\eta.\sin(\phi+\pi/2)=\xi.\sin\phi+\eta.\cos\phi,\ \text{etc.};$$

6.4. *General Cartesian transformation.*

This, of course, combines § 6.2.1. and § 6.2.2.

7. *Fundamental trigonometric theorems.*

Reverting to 3.2.2. :

7.1. *Opposite angles.*†

If P_1P_2 is a chord of the circle, in the Standard Diagram of Trigonometry, at right angles to the line OA, then M_1, M_2 coincide and $ON_1+ON_2=0$. Hence

$$\cos(-\theta)=\cos\theta,\ \text{and}\ \sin(-\theta)=ON_2:OB=-(ON_1:OB)=-\sin\theta,$$

are *general* theorems.

7.2. *Addition of positive right angle.*

If P_1, P_2 are points of the circle such that $\angle P_1OP_2$ is a positive right angle, then OP_2 is related to OB, OA' as OP_1 to OA, OB.

Hence $\cos(\theta+\pi/2)=(OM_2:OA)=-(OM_2:OA')=-(ON_1:OB)=-\sin\theta,$

and
$$\sin(\theta+\pi/2)=(ON_2:OB)=(OM_1:OA)=\cos\theta,$$

are *general* theorems.

7.3. *The general addition theorems.*

A special particular case (P on the unit circle) of the special particular case of § 6.3.2. ; but, again, a case for applying, not a general formula but the general method (of § 6.2.2.).

In the Standard Diagram, if $\angle AO\alpha$ is the ϕ-angle, and $\angle \alpha OP$ the ψ-angle, we get (by this method of resolution, or orthogonal projection),

$$\begin{matrix}\cos\\ \sin\end{matrix}(\phi+\psi)=\cos\psi.\begin{matrix}\cos\\ \sin\end{matrix}\phi+\sin\psi.\begin{matrix}\cos\\ \sin\end{matrix}(\phi+\pi/2)\ ;$$

and, thence,
$$\cos(\phi+\psi)=\cos\phi.\cos\psi-\sin\phi.\sin\psi$$
$$\sin(\phi+\psi)=\sin\phi.\cos.\psi+\cos\phi.\sin\psi$$

* It is worth noting that, here and in § 7—and generally in " elementary " theory (as distinguished from " analysis ")—there is no special virtue in using circular measure, except *convenience of printing*. Always, in such theory, the " argument " of the trigonometric function is the *angle* itself (rather than its " measure ") ; the " function " (*cos* or *sin*) being a *ratio* and, therefore, a " number ".

† *i.e.* two angles of which the sum = 0.

—these results being, of course, *quite general* and leading to all the standard related theory.*

8. *Conclusion.*

These notes cover most of the ground in question, and should serve to make clear the fundamental significance and importance of " sign ", in that field. Sign of *area* (related to the two opposite directions " normal " to the plane) also arises fundamentally in this context ; but, for even summary adequate treatment, would require more space than can be given to it here.

It may, perhaps, be suggested that the line of advance in mathematical training, about which Professor Neville is so deeply (and justifiably) concerned,† is that the *fundamentally right* methods be sought—regardless of every other consideration—over the whole field of what is (and will presumably remain) fundamental for groundwork training in this *absolutely* " *exact* " science : *precision* and *generality* bringing the reward of a higher order of *simplicity*, and of a great accompanying *beauty*.

There is some danger in the temptation to introduce, prematurely, subjects and methods which are properly appropriate to a later stage. (This whole question needs thoroughgoing investigation.) But the elimination of looseness, and essential inadequacy, and the promotion of accuracy and precision, and a proper dignity, can never be premature in the serious treatment of this great science ; and it is surely high time that this was generally recognised.

The foregoing article was written—in its original form—towards the end of the year 1942. It grew out of what was intended to be a Note, following upon Notes 1551, 1594 and 1595 (see §§ 4, 5 above—with the illustrative example, particularly well-chosen for Note 1594). Subsequent correspondence with the Editor—at a time of peculiarly difficult war-time communication (especially for him)—led to some considerable revision, which has not been completed until the middle of this year! Meanwhile, my mind had been running on certain questions of sign in elementary differential geometry—especially since the article on Curvature in XXVII, 274, p. 77 and the Note 1754 (XXVIII, 282, p. 185) appeared. I had meant to add a note about this ; but the article by Mr. Robson in XXX, 288, p. 1 (the last number of the *Gazette* which has reached me by the middle of August : showing sea-mails almost at their worst—in the transition period) covers the ground in question so thoroughly that nothing is left to be done but the dotting of some i's and the crossing of some t's. This I hope to do in a subsequent Note.

May I close this rather conversational note by paying a warm tribute to the astonishing continuance of the *Gazette* through the terrible years of *blitz*?

D. K. PICKEN.

Melbourne, Australia.

* This is the only satisfactory way of dealing with that standard theory, and that should be recognised once for all. It is set out carefully in Ch. V of my *Theory of Elementary Trigonometry* : now out of print ; there is a copy in the Math. Ass. Library. (At the time when that book was written (1910), I had not clearly realised the better alternative—of " resolutes "—to orthogonal projection.)

† *Vide* the urgent impatience in Note 1595 ; but, more especially, the Presidential Address, " The Food of the Gods " (*Gazette*, XIX, No. 232, February 1935, pp. 5–17).

GLEANINGS FAR AND NEAR.

1495. He was next year elected honorary president of the institution, an office of less than no emolument—since the holder was expected to come down handsome with a donation.—R. L. Stevenson, *The Wrong Box*, Chap. I (1889). [Per Prof. E. H. Neville.]

SPHERES AND CONICOIDS THROUGH TWENTY-FOUR POINTS.

By R. T. Robinson.

Preliminary. Figure 1 is described in detail in my article in the *Mathematical Gazette*, XIX, No. 236. It is based on a tetrahedron $ABCD$ and an arbitrary point D_2. These five points determine all the other points of the figure, and if the tetrahedron $ABCD$ is taken as the tetrahedron of reference and the coordinates of D_2 are $(\alpha_0, \beta_0, \gamma_0, \delta_0)$, the coordinates of A_1, B_1, C_1, D_1 are $(-\alpha_0, \beta_0, \gamma_0, \delta_0)$, $(\alpha_0, -\beta_0, \gamma_0, \delta_0)$, and so on. In what follows the tetrahedron $A_1B_1C_1D_1$ is called the associated tetrahedron.

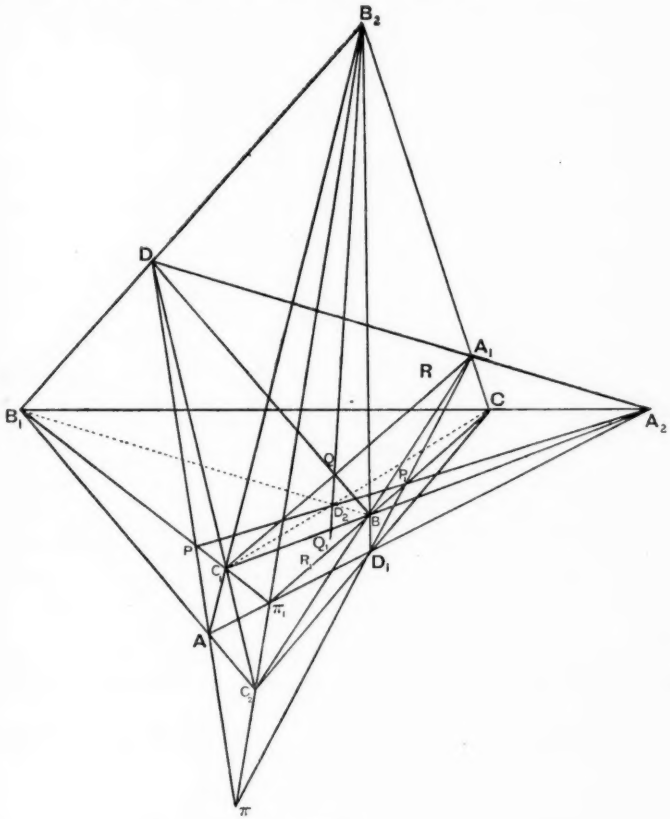


Fig. 1.

When the coordinates of D_2 are (R_1, R_2, R_3, R_4) or $(aef/A, dbf/B, cde/C, abc/D)$ where $R_1, R_2, \ldots$ are the radii of the circumcircles of the triangles $BCD, ACD, \ldots$, the associated tetrahedron and the tetrahedron $ABCD$ are

inscribed in the same sphere and the following relations hold :

(i) $V' : V = \Sigma(a_1e_1f_1) : \Sigma(aef)$, where V' and V are the volumes of the tetrahedra $A_1B_1C_1D_1$ and $ABCD$;

(ii) if the tetrahedron $ABCD$ is such that $ad = be = cf$, then in the tetrahedron $A_1B_1C_1D_1$ we have $a_1d_1 = b_1e_1 = c_1f_1$;

(iii) if the tetrahedron $ABCD$ is such that

$$a^2 + d^2 = b^2 + e^2 = c^2 + f^2,$$

then in the tetrahedron $A_1B_1C_1D_1$ a similar relation holds.

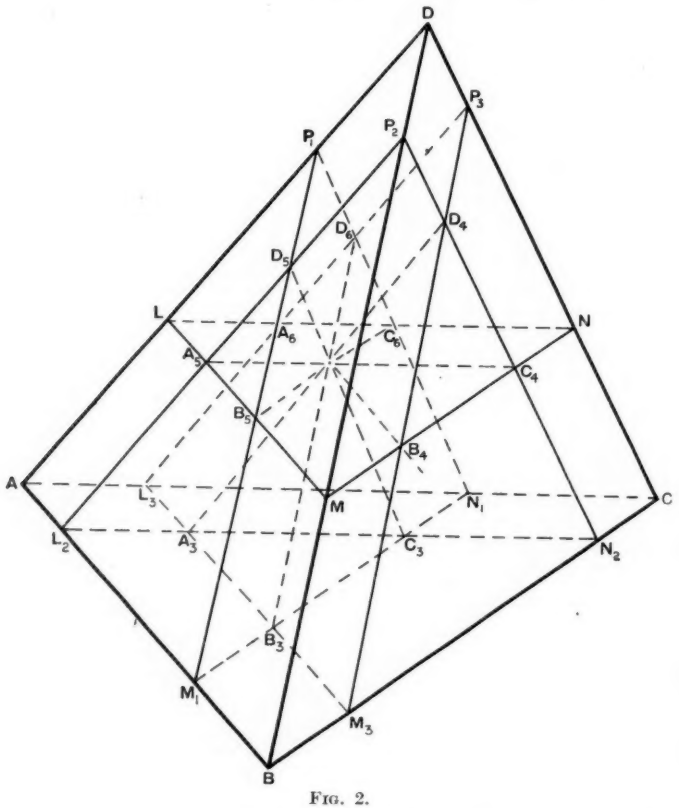


FIG. 2.

1. *If planes are drawn through D_2 parallel to the faces of the tetrahedron $ABCD$ each of the six edges of the tetrahedron will be cut in two points and there will be twelve points of intersection. And if planes are drawn through D_2 parallel to the faces of the associated tetrahedron $A_1B_1C_1D_1$ we get another set of twelve*

points on the edges of this tetrahedron. These twenty-four points lie on a sphere when the coordinates of D_2 are (R_1, R_2, R_3, R_4) and $ad = be = cf$.

Let the plane through D_2 parallel to ABC meet DA, DB, DC respectively in L, M, N. The equation of the plane is

$$\delta \,.\, \Sigma A\alpha_0 - \delta_0 \,.\, \Sigma A\alpha = 0.$$

We have the following sets of coordinates :

$$\begin{aligned} L &\quad (A\alpha_0 + B\beta_0 + C\gamma_0)/A,\ 0,\ 0,\ \delta_0 ; \\ M &\quad 0,\ (A\alpha_0 + B\beta_0 + C\gamma_0)/B,\ 0,\ \delta_0 ; \\ N &\quad 0,\ 0,\ (A\alpha_0 + B\beta_0 + C\gamma_0)/C,\ \delta_0. \end{aligned}$$

If the plane through D_2 parallel to BCD meets AD, AC, AB in P_1, N_1, M_1, we have the following sets of coordinates :

$$\begin{aligned} P_1 &\quad \alpha_0,\ 0,\ 0,\ (B\beta_0 + C\gamma_0 + D\delta_0)/D ; \\ N_1 &\quad \alpha_0,\ 0,\ (B\beta_0 + C\gamma_0 + D\delta_0)/C,\ 0 ; \\ M_1 &\quad \alpha_0,\ (B\beta_0 + C\gamma_0 + D\delta_0)/B,\ 0,\ 0. \end{aligned}$$

For the two points L and P_1 on AD,

$$\begin{aligned} DL \,.\, DP_1 &= d^2 \,.\, A\alpha_0(A\alpha_0 + B\beta_0 + C\gamma_0)/9V^2, \\ AL \,.\, AP_1 &= d^2 \,.\, D\delta_0(B\beta_0 + C\gamma_0 + D\delta_0)/9V^2. \end{aligned}$$

Similarly for the points M and P_2 on BD,

$$\begin{aligned} DM \,.\, DP_2 &= e^2 \,.\, B\beta_0(A\alpha_0 + B\beta_0 + C\gamma_0)/9V^2, \\ BM \,.\, BP_2 &= e^2 \,.\, D\delta_0(A\alpha_0 + B\beta_0 + C\gamma_0)/9V^2, \text{ and so on.} \end{aligned}$$

These twelve points lie on a sphere if at the same time

$$\begin{aligned} DL \,.\, DP_1 &= DM \,.\, DP_2 = DN \,.\, DP_3, \\ AL \,.\, AP_1 &= \ldots\ldots, \\ &\ldots\ldots, \end{aligned}$$

that is, if

$$\begin{aligned} d^2A\alpha_0 &= e^2B\beta_0 = f^2C\gamma_0, \\ c^2B\beta_0 &= b^2C\gamma_0 = d^2D\delta_0, \\ c^2A\alpha_0 &= a^2C\gamma_0 = e^2D\delta_0, \\ b^2A\alpha_0 &= a^2B\beta_0 = f^2D\delta_0. \end{aligned}$$

These relations hold simultaneously if

$$ad = be = cf = \lambda, \text{ say, where } 576V^2\rho_1{}^2 = 3\lambda^4,$$

and α_0, β_0, γ_0, δ_0 are proportional to aef/A, dbf/B, cde/C, abc/D, that is, proportional to R_1, R_2, R_3, R_4.

2. The equation of the sphere through the four points L, M, N, N_1 is

$$\begin{aligned} \Sigma A\alpha \,.\, \{ & A\alpha \,.\, \lambda bcd(\Sigma aef - aef) + B\beta \,.\, \lambda ace(\Sigma aef - dbf) \\ & + C\gamma \,.\, \lambda abf(\Sigma aef - cde) + D\delta \,.\, \lambda def(\Sigma aef - abc)\} \\ & - (\Sigma aef)^2 \,.\, \Sigma c^2AB\alpha\beta = 0 \,. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)} \end{aligned}$$

This sphere passes through P_1, M_1. It can also be shown that if planes are drawn through D_2 parallel respectively to the planes ACD, ABD to meet the edges of the tetrahedron $ABCD$, these points of intersection also lie on the sphere. With the notation of Fig. 2, we have the sets of coordinates :

$$\begin{aligned} L_2 &\quad (A\alpha_0 + C\gamma_0 + D\delta_0)/A,\ 0,\ \ 0 ; \\ P_2 &\quad 0,\ \ 0,\ (A\alpha_0 + C\gamma_0 + D\delta_0)/D ; \\ N_2 &\quad 0,\ \ (A\alpha_0 + C\gamma_0 + D\delta_0)/C,\ 0 ; \\ L_3 &\quad (A\alpha_0 + B\beta_0 + D\delta_0)/A,\ 0,\ \gamma_0,\ 0 ; \\ M_3 &\quad 0,\ (A\alpha_0 + B\beta_0 + D\delta_0)/B,\ \gamma_0,\ 0 ; \\ P_3 &\quad 0,\ 0,\ \gamma_0,\ (A\alpha_0 + B\beta_0 + D\delta_0)/D. \end{aligned}$$

3. *If planes are drawn through D_2 parallel to the faces of the associated tetrahedron $A_1B_1C_1D_1$ to meet the edges of this tetrahedron, the points of intersection will lie on the same sphere.*

To prove this in the case of the point determined by the intersection of A_1D_1 with the plane through D_2 drawn parallel to the plane $B_1C_1D_1$, that is, parallel to $-\alpha/\alpha_0+\beta/\beta_0+\gamma/\gamma_0+\delta/\delta_0=0$, we proceed as follows: the equation of this plane is

$$-\frac{\alpha}{\alpha_0}+\frac{\beta}{\beta_0}+\frac{\gamma}{\gamma_0}+\frac{\delta}{\delta_0}-\frac{2\Sigma A\alpha}{\Sigma A\alpha_0}=0;$$

and since the coordinates of A_1, D_1 are respectively

$$(-\alpha,\ \beta_0,\ \gamma_0,\ \delta_0),\ (\alpha_0,\ \beta_0,\ \gamma_0-\delta_0),$$

the coordinates of any point on A_1D_1 are

$$\alpha_0(-1+k),\ \beta_0(1+k),\ \gamma_0(1+k),\ \delta_0(1-k),$$

and this point will lie on the plane if

$$k=(\Sigma A\alpha_0+2A\alpha_0)/(\Sigma A\alpha_0-2D\delta_0).$$

The coordinates of the point P_1' are

$$\alpha_0(A\alpha_0+D\delta_0),\ \beta_0(2A\alpha_0+B\beta_0+C\gamma_0),\ \gamma_0(2A\alpha_0+B\beta_0+C\gamma_0),\ -\delta_0(A\alpha_0+D\delta_0).$$

If these are substituted for α, β, γ, δ in the equation (i) of the sphere, we get, putting $\alpha_0=\theta aef/A$, $\beta_0=\theta dbf/B$, ...,

$$\Sigma A\alpha=\theta^2\,.\,\Sigma aef\,.\,(\Sigma aef-2abc),$$

also

$$\Sigma\left[\frac{A\alpha\lambda bcd}{(\Sigma aef)^2}(\Sigma aef-aef)\right]=\frac{\lambda^4\theta^2}{\Sigma aef}(\Sigma aef+2aef),$$

and

$$\Sigma c^2AB\alpha\beta=\lambda^4\theta^4(\Sigma aef+2aef)(\Sigma aef-2abc).$$

Thus the sphere whose equation is (i) passes through this point, and in a similar manner it can be shown to pass through the other eleven points on the edges of the associated tetrahedron $A_1B_1C_1D_1$.

4. The equation of the radical plane of this sphere and the sphere $ABCD$ is

$$\Sigma aef\,.\,\Sigma(A\alpha\,.\,bcd)-\lambda^3\,.\,\Sigma A\alpha=0.$$

Hence the radical plane is parallel to the plane whose equation is

$$\Sigma(A\alpha bcd)=0 \quad \text{or} \quad \alpha/R_1+\beta/R_2+\gamma/R_3+\delta/R_4=0;$$

that is, the radical plane is parallel to the polar of D_2 with respect to the sphere $ABCD$ and is perpendicular to the straight line $O_1\Omega$ where O_1 and Ω are the centres of the sphere $ABCD$ and the sphere whose equation is (i), and O_1, Ω, D_2 are collinear.

5. *To find the coordinates of the centre Ω of this sphere.*

When the equation of a sphere is $\Sigma l\alpha\,.\,\Sigma A\alpha-\Sigma c^2AB\alpha\beta=0$ the coordinates of its centre are given by equations of the type

$$\alpha=(-l+m\cos AB+n\cos AC+p\cos AD)/6V\ +\ 3V_1/A.$$

For the sphere through twenty-four points:

$$\begin{aligned}&-l+m\cos AB+n\cos AC+p\cos AD\\&=[-A\lambda bcd(\Sigma aef-aef)+B\lambda ace(\Sigma aef-dbf)\cos AB\\&\qquad+C\lambda abf(\Sigma aef-cde)\cos AC+D\lambda def(\Sigma aef-abc)\cos AD]/(\Sigma aef)^3)\end{aligned}$$

$$= \lambda[- Abcd + Bace \cos AB + Cabf \cos AC + Ddef \cos AD]/(\Sigma aef)$$
$$= \lambda[- A^2b^2c^2d^2 + \lambda^2(c^2AB \cos AB + b^2AC \cos AC + d^2AD \cos AD)]/A \,.\, bcd\,\Sigma aef$$
$$= \lambda[- 16A^2b^2c^2d^2 + \lambda^2 \,.\, 288(V^2 - VV_1)]/16Abcd\,\Sigma aef$$
$$= \lambda^3(aef\,\Sigma aef - \Sigma a^2e^2f^2)/16Abcd\,\Sigma aef$$
$$= aef(aef\,\Sigma aef - \Sigma a^2e^2f^2)/16A\,\Sigma aef.$$

Hence

$$\alpha = \frac{V}{\Sigma aef} \cdot \frac{aef}{A} + 2\,\frac{V_1}{A},$$

with similar expressions for the other coordinates. From this we see that *the centre Ω lies on O_1D_2 and trisects it*, that is, $O_1\Omega = \frac{1}{3}O_1D_2$.

6. *To find O_1D_2.*

The coordinates of D_2 are $\theta aef/A, \ldots,$ with $\theta = 3V/\Sigma aef$. Thus

$$-9V^2(O_1D_2)^2 = \Sigma c^2AB\left(\frac{\theta aef}{A} - \frac{3V_1}{A}\right)\left(\frac{\theta dbf}{B} - \frac{3V_2}{B}\right)$$
$$= 6\theta^2\lambda^4 - 3\theta\Sigma(c^2V_1\,dbf + c^2V_2\,aef) + 9V^2\rho_1^2,$$

and the second term on the right is equal to

$$\frac{9V}{\Sigma aef} \cdot \Sigma[aef(c^2V_2 + b^2V_3 + d^2V_4)]$$
$$= 9V[\Sigma(aef \,.\, 2V\rho_1^2)]/\Sigma aef$$
$$= 18V^2\rho_1^2.$$

Hence $(O_1D_2)^2 = \rho_1^2 - 6\lambda^4/(\Sigma aef)^2$

$$= \rho_1^2 - 2 \,.\, 576 \,.\, V^2\rho_1^2/(\Sigma aef)^2,$$

and $O_1\Omega^2 = \rho_1^2/9 \; - \; 128V^2\rho_1^2/(\Sigma aef)^2$.

7. *To find the radius of the twenty-four points sphere.*

If T is the length of the tangent from O_1 to the twenty-four points sphere,

$$9V^2T^2 = \frac{9\lambda^4}{288} - \frac{9V^2\lambda^4}{(\Sigma aef)^2} - 9V^2\rho_1^2,$$

where the coordinates of O_1 are substituted in (i), the equation to the sphere, and $\Sigma V_1 bcd = (\lambda^3\Sigma aef)/288V$.

Thus

$$O_1\Omega^2 - \rho^2 = \frac{\lambda^4}{288V^2} - \frac{\lambda^4}{(\Sigma aef)^2} - \rho_1^2,$$

and, recalling that $128V^2\rho_1^2 = \lambda^3$, and using § 6, we have

$$\rho^2 = 4\rho_1^2/9 \quad + \quad 64V^2\rho_1^2/(\Sigma aef)^2.$$

The points O_1, Ω and D_2 are the same for both tetrahedra $ABCD$ and $A_1B_1C_1D_1$, and when $A_1B_1C_1D_1$ is taken as tetrahedron of reference, we have

$$\rho^2 = 4\rho_1^2/9 \quad + \quad 64V_1^2\rho_1^2/(\Sigma a_1e_1f_1)^2,$$

where V_1 is the volume of $A_1B_1C_1D_1$, and hence

$$V : V_1 = (\Sigma aef) : (\Sigma a_1e_1f_1).$$

This may also be proved directly.

8. The section of this twenty-four points sphere by the plane ABC (Fig. 2) is a circle passing through the six points L_2, L_3, M_1, M_3, N_1, N_2. The planes through D_2 parallel to the faces of the tetrahedron $ABCD$ meet the plane

ABC in the sides of the triangle $A_3B_3C_3$: the sides of this triangle are therefore parallel to the sides of the triangle ABC. This circle through the six points is a Lemoine circle. The centre of perspective for the two triangles ABC, $A_3B_3C_3$ is the symmedian point whose trilinear coordinates are (a, b, c), for the quadriplanar coordinates of A_3 are

$$(aef + abc)/A,\ dbf/B,\ cde/C,\ 0,$$

and its trilinear coordinates are $ef + bc$, df, de : similarly the trilinear coordinates of B_3 are ef, $df + ac$, de, and of C_3 are ef, df, $de + ab$. The straight lines AA_3, BB_3, CC_3 meet in the centre of perspective whose coordinates are ef, df, de, that is, a, b, c, since $ad = be = cf$.

If this circle is a Lemoine circle its centre is the middle point of the straight line joining the centres of the circles ABC, $A_3B_3C_3$. This can be shown to be the case ; for the equation of the circle $A_3B_3C_3$ is

$$\lambda\{d\alpha(2\Sigma aef - 3aef) + e\beta(2\Sigma aef - 3dbf) + f\gamma(2\Sigma aef - 3cde)\}\Sigma a\alpha - (\Sigma aef)^2 \,.\, \Sigma a\beta\gamma = 0,$$

and the coordinates of its centre are

$$\alpha = 2R_4\lambda(-d + e\cos C + f\cos B)/\Sigma aef + R_4\cos A, \text{ etc.}$$

The equation of the circle which is the intersection of the twenty-four points sphere by the plane ABC or $\delta = 0$ is

$$\lambda\{d\alpha(\Sigma aef - aef) + e\beta(\Sigma aef - dbf) + f\gamma(\Sigma aef - cde)\}\Sigma a\alpha - (\Sigma aef)^2 \,.\, \Sigma a\beta\gamma = 0,$$

and the coordinates of its centre are

$$\alpha = R_4\lambda(-d + e\cos C + f\cos B)/\Sigma aef + R_4\cos A, \text{ etc.},$$

and this point is evidently the middle point of the straight line joining the centres of the circles ABC and $A_3B_3C_3$.

9. *This twenty-four points sphere is such that its intersection with each face of the tetrahedron $ABCD$ is a Lemoine circle and the lines joining A, B, C, D respectively to D_2 meet the opposite faces of the tetrahedron in the symmedian points of these faces, that is, in the points* $(0, a, e, f)$, $(d, 0, b, f)$, $(c, d, 0, e)$, $(a, b, c, 0)$. *The same results hold for the tetrahedron $A_1B_1C_1D_1$ and the point D_2.*

10. *Another twenty-four points sphere closely associated with the one just described can be obtained as follows.*

If in a tetrahedron $ABCD$ straight lines are drawn through D_2, the point whose coordinates are (R_1, R_2, R_3, R_4) parallel to the edges of the tetrahedron, we get six straight lines, each one of them meeting two of the faces of the tetrahedron (Fig. 2). Taking the lines through D_2,

the parallel to AD meets ABC in A_3 and BCD in D_4 ;
the parallel to BD meets ABC in B_3 and ACD in D_6 ;
the parallel to CD meets ABC in C_3 and ABD in D_5 ;
the parallel to AB meets BCD in B_4 and ACD in A_6 ;
the parallel to BC meets ADB in B_5 and ACD in C_6 ;
the parallel to AC meets ADB in A_5 and BCD in C_4.

In this way we get four tetrahedra $D_2A_3B_3C_3$, $D_2B_4C_4D_4$, $D_2A_5B_5D_5$, $D_2A_6C_6D_6$ with a common vertex D_2. Each tetrahedron is similar and similarly situated to the tetrahedron $ABCD$.

When the tetrahedron $ABCD$ is such that $ad = be = cf$ the twelve angular points of the bases of these four tetrahedra, that is, the points A_3, B_3, C_3 ; B_4, C_4, D_4 ; ... all lie on a sphere.

To prove this we can proceed as follows: the distance x between two points whose actual coordinates are $(\alpha_1, \beta_1, \gamma_1, \delta_1)$ $(\alpha_2, \beta_2, \gamma_2, \delta_2)$ is given by

$$-9V^2x^2 = \Sigma c^2 AB(\alpha_1 - \alpha_2)(\beta_1 - \beta_2).$$

Using this formula for the points D_2 and A_3 whose coordinates are respectively

$$\{aef/A, dbf/B, cde/C, abc/D\},\ \{(aef + abc)/A, dbf/B, cde/C, 0\},$$

we get $A_3D_2 = abc \,.\, d/\Sigma aef$. Otherwise we can see from the figure that $A_3D_2 : d$ is equal to the ratios of the δ's of D_2, D is therefore equal to $abc : \Sigma aef$. Every edge of the tetrahedron $D_2A_3B_3C_3$ is equal to the corresponding edge of the tetrahedron $ABCD$ multiplied by $abc/\Sigma aef$. Similarly each edge of the tetrahedron $D_2B_4C_4D_4$ is equal to the corresponding edge of the tetrahedron $ABCD$ multiplied by $aef/\Sigma aef$. Hence $D_2D_4 = aef \,.\, d/\Sigma aef$, and hence

$$A_3D_2 \,.\, B_3D_4 = abc \,.\, aef \,.\, d^2/(\Sigma aef)^2 = \lambda^4/(\Sigma aef)^2.$$

Similarly the products $B_3D_2 \,.\, D_2D_6$, $C_3D_2 \,.\, D_2D_5$, $B_4D_2 \,.\, D_2A_6$, $A_5D_2 \,.\, D_2C_4$, $A_6D_2 \,.\, D_2B_4$ are equal and have the same value. Thus the twelve points A_3, B_3, C_3, ; B_4, C_4, D_4 ; ... all lie on a sphere.

The equation of the sphere which passes through these twelve points is

$$\begin{aligned}\lambda \,.\, \Sigma A\alpha\{&A\alpha \,.\, bcd(2\Sigma aef - 3aef) + B\beta \,.\, ace(2\Sigma aef - 3dbf)\\ &+ C\gamma \,.\, abf(2\Sigma aef - 3cde) + D\delta \,.\, def(2\Sigma aef - 3abc)\}\\ &- (\Sigma aef)^2 \Sigma c^2 AB\alpha\beta = 0\ ;\end{aligned}$$

the coordinates $(\alpha, \beta, \gamma, \delta)$ of its centre Ω_1 are such that $2\alpha_1 = \alpha + 3V_1/A$, $2\beta_1 = \beta + 3V_2/B$, etc., where $(\alpha_1, \beta_1, \gamma_1, \delta_1)$ are the coordinates of the centre Ω of the twenty-four points sphere first described.

Thus the centre Ω of the first twenty-four points sphere is the middle point of the line joining O_1, the centre of the sphere $ABCD$, to Ω_1, the centre of the second twenty-four points sphere; hence $O_1\Omega = \Omega\Omega_1 = \Omega_1D_2$.

The radius ρ of the second twenty-four points sphere is given by

$$\rho^2 - D_2\Omega_1{}^2 = \lambda^4/(\Sigma aef)^2 \quad \text{and} \quad D_2\Omega_1{}^2 = \tfrac{1}{9}D_2O_1{}^2 = \tfrac{1}{9}\rho_1{}^2 - 128V^2\rho_1{}^2/(\Sigma aef)^2.$$

Thus

$$\rho^2 = \tfrac{1}{9}\rho_1{}^2 + 64V^2\rho_1{}^2/(\Sigma aef)^2.$$

11. If the tetrahedron $ABCD$ is such that $ad = be = cf$ it can be shown that in the associated tetrahedron $A_1B_1C_1D_1$, $a_1d_1 = b_1e_1 = c_1f_1$, when this tetrahedron is derived from $ABCD$ and D_2 in the manner indicated. When these two tetrahedra have the same circumscribing sphere and the coordinates of D_2 are (R_1, R_2, R_3, R_4), if lines are drawn through D_2 parallel to the edges of the tetrahedron $A_1B_1C_1D_1$, each line intersecting two faces of the tetrahedron, we get twelve points of intersection which will lie on a sphere and the centre of the sphere will divide O_1D_2 in the ratio 2 : 1 ; also the square of the radius is

$$\tfrac{1}{9}\rho_1{}^2 + 64V_1{}^2\rho_1{}^2/(\Sigma a_1e_1f_1)^2,$$

or, as before,

$$\tfrac{1}{9}\rho_1{}^2 + 64V^2\rho_1{}^2/(\Sigma aef)^2.$$

Therefore the second sphere also passes through twenty-four points.

12. This can be proved otherwise as follows: the coordinates of D_2, A_1, B_1 are $(\alpha_0, \beta_0, \gamma_0, \delta_0)$, $(-\alpha_0, \beta_0, \gamma_0, \delta_0)$, $(\alpha_0, -\beta_0, \gamma_0, \delta_0)$. The equation of the line through D_2 parallel to A_1B_1 is

$$\frac{1}{l}\left(\alpha - \frac{3V\alpha_0}{\Sigma A\alpha_0}\right) = \frac{1}{m}\left(\beta - \frac{3V\beta_0}{\Sigma A\alpha_0}\right) = \frac{1}{n}\left(\gamma - \frac{3V\gamma_0}{\Sigma A\alpha_0}\right) = \frac{1}{p}\left(\delta - \frac{3V\delta_0}{\Sigma A\alpha_0}\right) = \theta, \text{ say,}$$

where $l : m : n : p = -\alpha_0(C\gamma_0 + D\delta_0) : \beta_0(C\gamma_0 + D\delta_0) : \gamma_0(A\alpha_0 - B\beta_0) : \delta_0(A\alpha_0 - B\beta_0)$.

For the intersection of this line with the plane $B_1C_1D_1$, whose equation is

$$-\alpha/\alpha_0+\beta/\beta_0+\gamma/\gamma_0+\delta/\delta_0=0,$$

we get $\theta=-3VK/(\Sigma A\alpha_0)$, where $1/K=A\alpha_0-B\beta_0+C\gamma_0+D\delta_0$, and the coordinates of the point P where D_2P, parallel to A_1B_1, meets the plane $B_1C_1D_1$ are given by

$$\alpha=3V\alpha_0\{1+K(C\gamma_0+D\delta_0)\}/\Sigma A\alpha_0,$$
$$\beta=3V\beta_0\{1-K(C\gamma_0+D\delta_0)\}/\Sigma A\alpha_0,$$
$$\gamma=3V\gamma_0\{1-K(A\alpha_0-B\beta_0)\}/\Sigma A\alpha_0,$$
$$\delta=3V\delta_0\{1-K(A\alpha_0-B\beta_0)\}/\Sigma A\alpha_0,$$

and the coordinates of D_2 are $3V\alpha_0/\Sigma A\alpha_0$, $3V\beta_0/\Sigma A\alpha_0$, ...

Thus $-9V^2\,.\,D_2P^2=9V^2K^2G/(\Sigma A\alpha_0)^2$, where

$$G=\{-c^2AB\alpha_0\beta_0(C\gamma_0+D\delta_0)^2-b^2AC\alpha_0\gamma_0(C\gamma_0+D\delta_0)(A\alpha_0-B\beta_0)$$
$$-d^2AD\alpha_0\delta_0(C\gamma_0+D\delta_0)(A\alpha_0-B\beta_0)+a^2BC\beta_0\gamma_0(C\gamma_0+D\delta_0)(A\alpha_0-B\beta_0)$$
$$+e^2BD\beta_0\delta_0(C\gamma_0+D\delta_0)(A\alpha_0-B\beta_0)+f^2CD\gamma_0\delta_0(A\alpha_0-B\beta_0)^2\}.$$

Putting $\alpha_0=aef/A$, $\beta_0=dbf/B$, ... , this becomes

$$D_2P^2=-\lambda^4\{(aef-dbf)^2-(cde+abc)^2\}/(aef-dbf+cde+abc)^2(\Sigma aef)^2$$
$$=\lambda^4(-aef+dbf+cde+abc)/(aef-dbf+cde+abc)(\Sigma aef)^2.$$

If D_2P meets $A_1C_1D_1$ in Q we get, in the same way,

$$D_2Q^2=\lambda_4(aef-dbf+cde+abc)/(-aef+dbf+cde+abc)(\Sigma aef)^2,$$

whence $$D_2P\,.\,D_2Q=\lambda^4/(\Sigma aef)^2.$$

Thus the points P and Q lie on the sphere $A_3B_3C_3B_4$ ... .

Similarly, if we draw parallels through D_2 to the other edges of the tetrahedron $A_1B_1C_1D_1$, the points of intersection of these lines with the faces will lie on the same sphere. Incidentally we see that $D_2P\,.\,D_2Q=\lambda_1{}^4/(\Sigma a_1e_1f_1)^2$, and hence

$$\lambda_1{}^2/\Sigma a_1e_1f_1=\lambda^2/\Sigma aef,$$

that is, $$V_1/\Sigma a_1e_1f_1=V/\Sigma aef.$$

13. The twelve points $A_3B_3C_3B_4$ ... can be arranged in four sets of three, $A_3B_3C_3$; $B_4C_4D_4$; $A_5B_5D_5$; $A_6C_6D_6$, and the sides of these triangles are parallel to the sides of the triangles ABC, BCD, ABD, ACD. Each of the twelve sides of the triangles intersects two of the other sides in points which lie on the edges of the tetrahedron and these points are points on the first twenty-four-points sphere.

The coordinates of these twelve points are as follows :

B_3	aef/A,	$(dbf+abc)/B$,	cde/C,	0 ;
D_3	aef/A,	0,	cde/C,	$(abc+dbf)/D$;
C_3	aef/A,	dbf/B,	$(cde+abc)/C$,	0 ;
D_5	aef/A,	dbf/B,	0,	$(cde+abc)/D$;
C_4	0,	dbf/B,	$(aef+cde)/C$,	abc/D ;
A_5	$(aef+cde)/A$,	dbf/B,	0,	abc/D ;
A_6	$(aef+dbf)/A$,	0,	cde/C,	abc/D ;
B_4	0,	$(aef+dbf)/B$,	cde/C,	abc/D ;
B_5	aef/A,	$(dbf+cde)/B$,	0,	abc/D ;

C_6 $aef/A,\ 0,\ (dbf+cde)/C,\ abc/D$;
A_3 $(aef+abc)/A,\ dbf/B,\ cde/C,\ 0$;
D_4 $0,\ dbf/B,\ cde/C,\ (aef+abc)/D$.

14. The first twenty-four-points sphere meets BD in M, BA in M_1, BC in M_3 (see Fig. 2), where $BM=(abc/\Sigma aef)e$, $BM_3=(cde/\Sigma aef)a$, $BM_1=(aef/\Sigma aef)c$, and $MM_1=M_1M_3=MM_3=\lambda^2/\Sigma aef$. Hence *the points M, M_1, M_3 on the sphere form an equilateral triangle.*

The equation of the plane MM_1M_3 is

$$(A\alpha bcd+C\gamma abf+D\delta def)\ .\ (\Sigma aef)-\lambda^3\Sigma A\alpha=0,$$

and the plane is parallel to the tangent plane at B to the sphere $ABCD$, for the equation of this tangent plane is

$$c^2A\alpha+a^2C\gamma+e^2D\delta=0,$$

or

$$A\alpha\ .\ bcd+C\gamma\ .\ abf+D\delta\ .\ def=0.$$

The length of the perpendicular from Ω (the centre of the first twenty-four-points sphere) on to the plane MM_1M_3 is $2\rho_1/3$, for the coordinates of Ω are

$$\frac{aef}{A}\cdot\frac{V}{\Sigma aef}+\frac{2V_1}{A}:\frac{dbf}{B}\cdot\frac{V}{\Sigma aef}+\frac{2V_2}{B}:\ldots.$$

Hence the length of the perpendicular from Ω to this plane is

$$3\lambda^3dbf/144V\sqrt{H},$$

using the relations in this type of tetrahedron :

(i) $\Sigma V_1,\ bcd=\lambda^3(\Sigma aef)/288V$;

(ii) $288VV_2=dbf(\Sigma aef)-3d^2b^2f^2$;

and the number H under the square root sign is

$$\begin{aligned}
&A^2b^2c^2d^2+C^2a^2b^2f^2+D^2d^2e^2f^2-2\lambda^2(b^2AC\cos AC+d^2AD\cos AD+f^2CD\cos CD)\\
&\quad=\lambda^2\ .\ 18(V^2-VV_1)-\tfrac{1}{16}\lambda^2(aef\ .\ \Sigma aef-\Sigma a^2e^2f^2)\\
&\qquad+\lambda^2\ .\ 18(V^2-VV_3)-\tfrac{1}{16}\lambda^2(cde\Sigma aef-\Sigma a^2e^2f^2)\\
&\qquad+\lambda^2\ .\ 18(V^2-VV_4)-\tfrac{1}{16}\lambda^2(abc\Sigma aef-a^2e^2f^2)-\lambda^2\ .\ 18(V^2+2VV_2)\\
&\quad=\lambda^2(18V^2-18VV_2)-\tfrac{1}{16}\lambda^2\{(\Sigma aef)^2-3\Sigma a^2e^2f^2-dbf\ .\ \Sigma aef\}\\
&\quad=\tfrac{1}{16}\lambda^2\ .\ dbf\ .\ \Sigma aef-\lambda^2\ .\ 18VV_2,\text{ since }288V^2=(\Sigma aef)^2-3\Sigma a^2e^2f^2,\\
&\quad=\tfrac{3}{16}\lambda^2\ .\ d^2b^2f^2,\text{ since }288VV_2=dbf(\Sigma aef)-3d^2b^2f^2.
\end{aligned}$$

Hence the length of the perpendicular from Ω to the plane MM_1M_3 is

$$\lambda^2/12V\sqrt{3}=2\rho_1/3.$$

15. *Similarly, the length of each perpendicular from Ω, the centre of the first twenty-four-points sphere, to the planes LL_2L_3, NN_1N_2, $P_1P_2P_3$ is $2\rho_1/3$; therefore these planes and MM_1M_3 all touch a sphere centre Ω and radius $2\rho_1/3$: these planes are sections of the tetrahedron $ABCD$ parallel respectively to the tangent planes at A, B, C, D to the sphere $ABCD$. The triangles LL_2L_3, MM_1M_3, ..., are equal equilateral triangles, the lengths of the sides being $\lambda^2/(\Sigma aef)$.*

Hence the square of the radius of the first twenty-four-points sphere is

$$\begin{aligned}
&4\rho_1^2/9 &+\ &(\text{radius of circle } LL_2L_3)^2\\
&=4\rho_1^2/9 &+\ &\lambda^4/3(\Sigma aef)^2\\
&=4\rho_1^2/9 &+\ &64V^2\rho_1^2/(\Sigma aef)^2,
\end{aligned}$$

as previously found.

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16. That the twelve points $L, L_2, L_3 \ldots$ lie on a sphere can be shown in this way :

If the plane ABC is met by DD_2 in K_4, then K_4 is the symmedian point of the triangle ABC, and the two triangles ABC, $A_3B_3C_3$ are in perspective from K_4. Therefore L_2, L_3, M_1, M_3, N_1, N_2 lie on a Lemoine circle. Similarly, in the triangle BCD the points M, M_3, N, N_2, P_2, P_3 lie on a circle, and so on. These four circles taken in pairs have—each pair—two points common, and are sections of the same sphere.

The same results hold for the associated tetrahedron $A_1B_1C_1D_1$ and the twelve points L', L_2', L_3' ... corresponding to L, L_2, $L_3 \ldots$. For example, the planes $L'L_2'L_3'$, ... touch a sphere centre Ω, where $O_1\Omega = \frac{1}{3}O_1D_2$ and radius $2\rho_1/3$, and this plane is parallel to the tangent plane to the sphere at A_1, and so on. Also the sides of the triangles $L'L_2'L_3'$ are each equal to $\lambda^2/(\Sigma aef)$.

17. *The twelve points $L, L_2, L_3 \ldots$ can be grouped in four triangles $M_1N_1P_1$; $L_2N_2P_2$; $L_3M_3P_3$; LMN, each plane passing through D_2 and having D_2 a symmedian point.*

If lines are drawn through D_2 parallel to the edges of the tetrahedron $ABCD$ meeting two faces of the tetrahedron, it can be seen from Fig. 2 that the twelve points lie on another sphere.

Taking, for example, the triangle LMN, the straight lines through D_1 parallel to BC, CA, AB, that is, to MN, NL, LM, are B_5C_6, A_5C_4, A_6B_4, and the six points B_5, C_6, A_5, C_4, A_6, B_4 lie on a circle. In the same way, from the triangle $M_1N_1P_1$, the six points B_5, C_6, C_3, D_5, B_3, D_6 lie on a circle and so on for the other two triangles $L_2N_2P_2$ and $L_3M_3P_3$. These four circles taken in pairs have two points common and they lie in different planes; therefore they are sections of the same sphere.

It can also be shown that A_3, A_5, A_6 are the vertices of an equilateral triangle whose sides are of length $\lambda^2/(\Sigma aef)$, and the same also holds for the triangles $B_3B_4B_5$, $C_3C_4C_6$, $D_4D_5D_6$.

18. The planes of these sets of points are parallel to the tangent planes at A, B, C, D to the sphere $ABCD$. The plane of the triangle $B_3B_4B_5$, for example, is parallel to the plane $c^2A\alpha + a^2C\gamma + e^2D\delta = 0$, which is the tangent plane at B, and its equation is

$$(A\alpha \,.\, bcd + C\gamma \,.\, abf + D\delta \,.\, def)(\Sigma aef) - 2\lambda^3(\Sigma A\alpha) = 0.$$

It can be shown that the perpendicular from Ω_1, the centre of the second twenty-four-points sphere to the plane $B_3B_4B_5$, is equal to $\rho_1/3$, and hence the square of the radius of the second twenty-four-points sphere is

$$\begin{aligned} &\rho_1^2/9 \quad + \quad (\text{radius of circle } B_3B_4B_5)^2 \\ &= \rho_1^2/9 \quad + \quad \lambda^4/3(\Sigma aef)^2 \\ &= \rho_1^2/9 \quad + \quad 64V^2\rho_1^2/(\Sigma aef)^2, \end{aligned}$$

as previously found.

The planes $A_3A_5A_6$, $B_3B_4B_5$, ... thus touch a sphere centre Ω_1 and radius $\rho_1/3$, and the same result holds for the corresponding planes $A_3'A_5'A_6'$, $B_3'B_4'B_5'$, ... of the tetrahedron $A_1B_1C_1D_1$.

19. *To show that when a tetrahedron is such that $ad = be = cf$, any section PQR parallel to the tangent plane at a vertex to the circumscribing sphere is an equilateral triangle.*

The equation of a plane parallel to the tangent plane at D to the sphere $ABCD$ is

$$d^2A\alpha + e^2B\beta + f^2C\gamma + k(\Sigma A\alpha) = 0,$$

and where this meets AD we have $\beta=0$, $\gamma=0$. By this and similar arguments we obtain the sets of coordinates :

$$P \quad -3kV/d^2A, \quad 0, \quad 0, \quad 3V(k+d^2)/d^2D;$$
$$Q \quad 0, \quad -3kV/e^2B, \quad 0, \quad 3V(k+e^2)/e^2D;$$
$$R \quad 0, \quad 0, \quad -3kV/f^2C, \quad 3V(k+f^2)/f^2D.$$

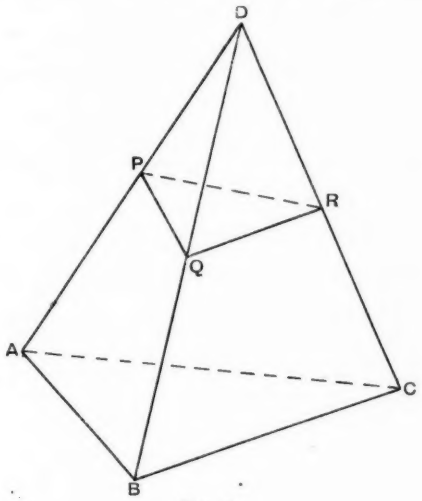


FIG. 3.

Hence $$-9V^2 \cdot PQ^2 = \Sigma c^2 AB(\alpha_1-\alpha_2)(\beta_1-\beta_2)$$
$$= -9V^2 \cdot c^2k^2/d^2e^2,$$

the other terms cancelling out, and thus we can show that

$$PQ = kabc/\lambda^2 = QR = RP.$$

It can also be shown that the centre of the circle circumscribing PQR lies on the line DD_2, where D_2 is the point $(aef/A, dbf/B, \ldots)$.

20. *The general case.*

In any tetrahedron it can be proved that if through any point

$$D_2(\alpha_0, \beta_0, \gamma_0, \delta_0)$$

planes be drawn parallel to the faces of the tetrahedron, these planes will cut the edges of the tetrahedron in twelve points, and these points lie on a conicoid whose equation is

$$\{\alpha(\Sigma A\alpha_0 - A\alpha_0)/\alpha_0 + \beta(\Sigma A\alpha_0 - B\beta_0)/\beta_0 + \gamma(\Sigma A\alpha_0 - C\gamma_0)/\gamma_0 + \delta(\Sigma A\alpha_0 - D\delta_0)/\delta_0\} \cdot \Sigma A\alpha - \{\Sigma(\alpha\beta/\alpha_0\beta_0)\}\{\Sigma A\alpha_0\}^2 = 0. \quad \ldots\ldots(1)$$

If planes are drawn through D_2 parallel to the faces of the associated tetrahedron $A_1B_1C_1D_1$, these planes meet the edges of this tetrahedron in

another set of twelve points, and all these points lie on the same conicoid, whose equation has just been written. This can be proved in the same manner as in § 3. *These twenty-four points can be arranged in eight sets of three points,* $L L_2 L_3$; $M M_1 M_3$; ... $L'L'_2 L'_3$; $M'M'_1 M'_3$, ..., *and the eight planes are tangent planes to the conicoid*

$$(\Sigma A\alpha)\left\{\left(\Sigma\frac{\alpha}{\alpha_0}\right)(\Sigma A\alpha_0) - \tfrac{3}{2}\Sigma A\alpha\right\} - (\Sigma A\alpha_0)^2\,\Sigma\frac{\alpha\beta}{\alpha_0\beta_0} = 0,$$

LL_2L_3, for example, touching the conicoid at the point where the straight line AD_2 meets the plane, and similarly for the others. This conicoid and the conicoid through the twenty-four points have the same centre Ω and the same asymptotic cone.

Further, in any tetrahedron it can also be shown that if through any point $D_2(\alpha_0, \beta_0, \gamma_0, \delta_0)$ straight lines be drawn parallel to the edges of the tetrahedron, these straight lines will meet the faces of the tetrahedron in twelve points and these points lie on the conicoid whose equation is

$$(\Sigma A\alpha)\left\{\left(\Sigma\frac{\alpha}{\alpha_0}\right).\,2\Sigma A\alpha_0 - 3\Sigma A\alpha\right\} - (\Sigma A\alpha_0^2).\,\Sigma\frac{\alpha\beta}{\alpha_0\beta_0} = 0, \quad \text{.....(2)}$$

and if straight lines are drawn through D_2 parallel to the edges of the associated tetrahedron $A_1B_1C_1D_1$, these straight lines will meet the faces of the tetrahedron in twelve points which lie on the same conicoid whose equation has just been written. *These twenty-four points can be arranged in eight sets of three points* $A_3A_5A_6$; $B_3B_4B_5$; ...; A_3', A_5', A_6'; B_3', B_4', B_5'; ...; *and these eight planes are tangent planes to the conicoid*

$$(\Sigma A\alpha)\left\{\left(\Sigma\frac{\alpha}{\alpha_0}\right).\,2\Sigma A\alpha_0 - \tfrac{3}{2}\Sigma A\alpha\right\} - (\Sigma A\alpha_0)^2\Sigma\frac{\alpha\beta}{\alpha_0\beta_0} = 0.$$

The plane $A_3A_5A_6$, for example, touches the conicoid at the point where the straight line AD_2 meets the plane, and similarly for the other planes. This conicoid and the conicoid through the second set of twenty-four points have the same centre Ω_1 and the same asymptotic cone.

22. It can be shown that if O_1 is the centre of the conicoid $\Sigma(\alpha\beta/\alpha_0\beta_0) = 0$ and Ω, Ω_1 are the centres of the first and second of the two twenty-four-points conicoids then Ω, Ω_1 trisect O_1D_2 as before. When the coordinates of D_2 are (R_1, R_2, R_3, R_4) and $ad = be = cf$, the two conicoids (1) and (2) become the twenty-four-points spheres.

23. For verification purposes it may be noticed that if D_2 is the point $(\alpha_0, \beta_0, \gamma_0, \delta_0)$ we have the following sets of coordinates :

$$\begin{aligned}
A_3 &\quad \alpha_0 + D\delta_0/A,\ \beta_0,\ \gamma_0,\ 0\,;\\
A_5 &\quad \alpha_0 + C\gamma_0/A,\ \beta_0,\ 0,\ \delta_0\,;\\
A_6 &\quad \alpha_0 + B\beta_0/A,\ 0,\ \gamma_0,\ \delta_0\,;
\end{aligned}$$

and in the associated tetrahedron $A_1B_1C_1D_1$,

$$\begin{aligned}
A_3' &\quad \alpha_0(-A\alpha_0 + D\delta_0),\ \beta_0(B\beta_0 + C\gamma_0),\ \gamma_0(B\beta_0 + C\gamma_0),\\
&\qquad \delta_0(-A\alpha_0 + 2B\beta_0 + 2C\gamma_0 + D\delta_0)\,;\\
A_5' &\quad \alpha_0(-A\alpha_0 + C\gamma_0),\ \beta_0(B\beta_0 + D\delta_0),\ \gamma_0(-A\alpha_0 + 2B\beta_0 + C\gamma_0 + 2D\delta_0),\\
&\qquad \delta_0(B\beta_0 + D\delta_0)\,;\\
A_6' &\quad \alpha_0(-A\alpha_0 + B\beta_0),\ \beta_0(-A\alpha_0 + B\beta_0 + 2C\gamma_0 + 2D\delta_0),\ \gamma_0(C\gamma_0 + D\delta_0),\\
&\qquad \delta_0(C\gamma_0 + D\delta_0).
\end{aligned}$$

The coordinates of these points satisfy equation (2), and similarly the points of the three other sets satisfy the equation.

24. It will be found that the points of §§ 1, 2, 3, twenty-four in number, satisfy the equation (1).

That the locus of points on the edges of $ABCD$ should be shared by points on the edges of the associated tetrahedron $A_1B_1C_1D_1$ is not surprising when it is noted that the edges of the two tetrahedra are the diagonal lines of the six faces of a cuboidal solid (see Fig. 1). It can be seen from Fig. 2 that the 48 points are vertices of 8 parallelepipeds which have D_2 for a common vertex and also have for a vertex one of the points $A, B, C, D, A_1, B_1, C_1, D_1$.

R. T. R.

1496. For the purpose of the enquiry bricks are defined as walling units which (*a*) consist of any hard and durable inorganic substance, other than refractory substances employed specifically on account of this property, (*b*) are suitable for building and bonding, (*c*) are of such a shape, size and weight that they can be manipulated by one hand.—*Letters of G. S. Gordon*, 1902-1942. [Per Mr. C. E. Kemp.]

1497. Do a simple sum on an odd piece of paper. Take your age, multiply it by the number of hours each day you're on your feet, then multiply again by your weight, then—think what your shoes have to carry about! Tons of wear—just the job for —— shoes.—From an advertisement. [Per Dr. F. C. Powell.]

1498. A non-pedigree cow which cost its owner £30 may beat the world's annual milk yield record, at present held by the famous Cherry, with 4,164 gallons. Maule is the name of the cow. But the owner of Cherry, Mr. Frank Worth, of Wiltshire, will not mind if he loses her record because he owns Maule as well. Maule has yielded 4,000 gallons in just over 45 days ; she still gives five gallons a day.—*Evening News*, March 2, 1945. [Per Mr. D. H. Sadler.]

1499. The allied march into Germany appears to be more than ever a matter of mathematical progression governed as much by what has come to be known as logistics as by the enemy's ability to resist it.—*The Times*, April 7, 1945. [Per Dr. Bertha Jeffreys.]

1500. Newton wrote not only the *Principia*, but also a treatise on the topography of Hell. Up to this day we all hold beliefs which are not only incompatible with observable facts, but with facts actually observed by us. —A. Koestler, *The Yogi and the Commissar*, p. 121 (1945).

[The generalisation may be true, but what of the too-apt instance? If the reference is to the *Observations on Daniel and the Apocalypse*, the description has no validity whatever. Had the writer perhaps some dim recollection of *The Sacred Theory of the Earth*, by Newton's ingenious contemporary, Thomas Burnet?] [Per Prof. E. H. Neville.]

1501. " If anyone was to blame " for the possibility that the Spice Islands might turn out to belong to Spain, " it was greedy King John II " of Portugal, " who had been so urgent that Pope Alexander's line of demarcation "—the rival claims of Portugal and Spain to all lands yet to be discovered or conquered were settled by a papal division of the world into meridian-bounded hemispheres—" should be shifted further west on this side of the globe and therefore east on the other."—*Ferdinand Magellan*, by E. F. Benson, p. 76 (1929). [Per Prof. E. H. Neville.]

1502. *From a script.* The radius of curvature at a point of a plane curve represents the radius from the origin to that point of the circle which could be drawn with centre at the origin to include that point. [Per Dr. E. A. Maxwell.]

SUR LA GÉOMÉTRIE RÉCENTE DU TÉTRAÈDRE.

PAR VICTOR THÉBAULT.

Introduction. Dans son " Mémoire sur le tétraèdre (*Bulletin de l'Académie royale de Belgique*, 1884), J. Neuberg a donné des propriétés du point L dont les coordonnées normales sont proportionnelles aux rayons des cercles circonscrits aux triangles des faces d'un tétraèdre *isodynamique*. Nous avons étendu ces propriétés au point L défini de la même façon dans un tétraèdre quelconque $T \equiv ABCD$ (*Annales de la Société Scientifique de Bruxelles*, 1922, pp. 173-177). P. Delens lui a consacré aussi des développements mais par des méthodes assez difficiles à lire (*Mathesis*, 1937 et 1938).

Dans un but de vulgarisation, il n'est sans doute pas inutile de reprendre cette question par la géométrie élémentaire en y ajoutant d'ailleurs des compléments peut-être inédits.

1. **Notations.** Etant donné un tétraèdre $T \equiv ABCD$, soient $BC = a$, $DA = a'$, $CA = b$, $DB = b'$, $AB = c$, $DC = c'$ les longueurs des arêtes ; S_a, S_b, S_c, S_d les aires des triangles des faces BCD, CDA, DAB, ABC ; R_a, R_b, R_c, R_d les rayons des cercles circonscrits à ces triangles ; (O, R) la sphère circonscrite, de centre O et de rayon R ; A, B, C, D les angles de chacune des faces avec la sphère circonscrite ; V le volume.

2. **Sections antiparallèles du tétraèdre.** La section $A_bA_cA_d$, dont les sommets A_b, A_c, A_d sont sur les arêtes AB, AC, AD, sera dite antiparallèle si son plan est perpendiculaire au rayon AO de la sphère circonscrite.

Théorème. *Toutes les sections antiparallèles faites dans les angles trièdres A, B, C, D sont des triangles semblables.*

En effet, A_bA_c et BC, A_cA_d et CD, A_dA_b et DB sont antiparallèles, d'où

$$AA_b \cdot c = AA_c \cdot b = AA_d \cdot a' = \lambda_a,$$

et

$$A_bA_c/aa' = A_cA_d/cc' = A_dA_b/bb' = \lambda_a/bca'.$$

On a de même

$$B_aB_c/bb' = B_cB_d/cc' = B_dB_a/aa' = \lambda_b/cab',$$

et les triangles $A_bA_cA_d$, $B_cB_dB_a$ sont semblables. Si l'on prend

$$\lambda_a/bca' = \lambda_b/cab' = \lambda_c/abc' = \lambda_d/a'b'c' = \lambda,$$

les sections antiparallèles sont *égales* et à toute valeur de λ correspondent quatre sections antiparallèles.

3. **Lieu du centre du cercle inscrit dans les sections antiparallèles d'un même trièdre.** Soit w_a le centre du cercle inscrit au triangle $A_bA_cA_d$, r_a le rayon du cercle inscrit ; d_b, d_c, d_d les distances de w_a aux plans des faces CDA, DAB, ABC. On a

$$d_b = r_a \sin(A_cA_d),\ d_c = r_a \sin(A_dA_b),\ d_d = r_a \sin(A_bA_c).$$

Or, le plan $A_bA_cA_d$ étant perpendiculaire à AO, $(A_cA_d) = \frac{1}{2}\pi - \angle OAO_b$, O_b étant le centre du cercle circonscrit au triangle CDA. Donc

$$R_b = R \cos OAO_b,$$

et, enfin,

$$d_b = r_a \cdot R_b/R,\ \ d_c = r_a \cdot R_c/R,\ \ d_d = r_a \cdot R_d/R.$$

Le lieu du centre w_a du cercle inscrit au triangle $A_bA_cA_d$ est la droite Δ_a (*symédiane* de T), définie par la relation

$$d_b/R_b = d_c/R_c = d_d/R_d.$$

4. **Second point de Lemoine L et angle du Brocard.** La symédiane Δ_a et ses analogues Δ_b, Δ_c, Δ_d concourent visiblement en un point L dont les distances aux plans des faces BCD, CDA, DAB, ABC, sont proportionnelles aux rayons R_a, R_b, R_c, R_d des cercles circonscrits aux triangles de ces faces (*second* point de Lemoine). Les coordonnées normales du point L sont donc données par les relations

$$d_a/R_a = d_b/R_b = d_c/R_c = d_d/R_d = 3V/\Sigma R_a S_a = 12V/\Sigma ab'c',$$

ou, en posant $12V \cot\theta = \Sigma ab'c'$,

$$d_a/R_a = d_b/R_b = d_c/R_c = d_d/R_d = \tan\theta, \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Les projections orthogonales du point L sur les *axes* des cercles (O_i, R_i), $(i = a, b, c, d)$ sont les sommets de quatre cônes de révolution semblables entre eux ayant même angle θ à la base (angle de Brocard).

5. **Sphères de Tucker. Théorème.** *Les cercles inscrits dans quatre sections antiparallèles égales sont sur une même sphère* (Ω) *dont le centre* Ω *est sur la droite* OL *joignant le centre de la sphère circonscrite au second point de Lemoine.*

Considerons, par exemple, deux sections antiparallèles égales des trièdres B et C ; w_b et w_c sont les centres des cercles inscrits dans ces sections. Soient L_a, w_b', w_c' les projections orthogonales des points L, w_b, w_c sur le plan de la face BCD. On a d'ailleurs, si r est le rayon commun des cercles inscrits égaux

$$w_b w_b' = w_c w_c' = r \,.\, R_a/R.$$

La droite $w_b w_c$ est parallèle à l'arête BC, et

$$Lw_b/LB = Lw_c/LC = w_b w_c/BC.$$

Si donc on mène par w_b et w_c des parallèles aux rayons OB et OC de la sphère circonscrite à T, ces parallèles rencontrent LO au même point Ω et, comme $OB = OC = R$, $\Omega w_b = \Omega w_c = l$.

Les cercles inscrits (w_i) sont donc sur une sphère de centre Ω et de rayon $\sqrt{(r^2 + l^2)}$. (Sphère de Tucker.)

Réciproquement, à tout point Ω de LO correspond une sphère (Ω) passant par les cercles inscrits à quatre sections antiparallèles égales.

Posons $L\Omega/LO = m$. On a successivement

$$L\Omega/LO = m = Lw_b/LB = (LB - Bw_b)/LB = 1 - Bw_b/LB.$$

Or,
$$Bw_b/LB = 1 - m = w_b w_b'/LL_a = w_b w_b'/R_a \tan\theta.$$

Donc
$$w_b w_b' = (1-m) R_a \tan\theta = r \,.\, R_a/R, \text{ et } r = (1-m) R \tan\theta.$$

Le rayon de la sphère (Ω) de Tucker est donc donné par la relation

$$\rho^2 = (1-m)^2 R^2 \tan^2\theta + (\Omega w_b)^2 = (1-m)^2 R^2 \tan^2\theta + m^2 R^2. \quad \ldots\ldots\ldots(2)$$

Seconde sphère de Lemoine (L) *(ou des cosinus).* En particulier, si $m = 0$, $\rho = R \tan\theta$. Il s'agit alors de la seconde sphère de Lemoine (L), de centre L, pour laquelle les quatres sections antiparallèles égales ont le point L comme centre commun de leurs cercles inscrits. Cette sphère rencontre le plan de la face BCD suivant un cercle dont le carré du rayon

$$\sigma^2 = \rho^2 - (LL_a)^2 = R^2 \tan^2\theta - R_a{}^2 \tan^2\theta = OO_a{}^2 \tan^2\theta.$$

Dès lors,
$$\sigma/\rho = OO_a \tan\theta/R \tan\theta = OO_a/R = R \cos A.$$

Première sphère de Lemoine. Si $m = \frac{1}{2}$, $\rho = \frac{1}{2} R$ séc θ. Le centre de cette sphère est au milieu de LO.

6. Surface enveloppe des sphères de Tucker (Ω). Prenons pour origine le point L, pour axe la droite LO et posons $LO = d$.

L'equation d'une sphère (Ω) sera

$$(x - md)^2 + y^2 + z^2 = (1 - m)^2 R^2 \tan^2\theta + m^2 R^2.$$

L'enveloppe de cette sphère a donc pour équation

$$\{(x^2 + y^2 + z^2) - R^2 \tan^2\theta\}\,(d^2 - R^2 \sec^2\theta) + (dx - R^2 \tan^2\theta)^2 = 0.$$

C'est un ellipsoïde de révolution (E) autour de LO (Ellipsoïde de Brocard).

Théorème. *L'ellipsoïde (E) est inscrit au tétraèdre T.*

Ses foyers F_1 et F_2 sont conjugués harmoniques par rapport aux points O et L. Le produit des distances de F_1 et F_2 aux plans des faces de T est égal à $R^2 \sin^2\theta$.

La relation (1) montre déjà qu'il y a sur la droite OL deux points F_1 et F_2 centres de deux sphères (Ω) de rayon nul. Ces points sont les foyers de l'ellipsoïde (E) ; comme ils sont imaginaires, l'ellipsoïde (E) est aplati. De plus, la relation (1) donne

$$m^2 - 2m \sin^2\theta + \sin^2\theta = 0.$$

Comme $x = md$, l'équation aux abscisses des points F_1 et F_2 est

$$(x^2/d^2) \quad - \quad 2(x/d)\sin^2\theta \quad + \quad \sin^2\theta = 0\ ;$$

d'où $\qquad 1/x_1 \quad + \quad 1/x_2 = 2/d, \qquad$ ou $\qquad 2/LO = 1/LF_1 \quad + \quad 1/LF_2.$

Enfin, si Ω_a est la projection orthogonale du centre Ω de la sphère (Ω) sur le plan BCD, on a

$$m = (\Omega\Omega_a - LL_a)/(OO_a - LL_a)\ ;$$

d'où

$$(\Omega\Omega_a - LL_a)^2 - 2\Omega\Omega_a(OO_a - LL_a)\sin^2\theta + (OO_a - LL_a)^2 \sin^2\theta = 0. \quad \ldots.(3)$$

Or, si l'on pose $R_a = R \sin A$,

$$LL_a = R \tan\theta \sin A, \quad OO_a = R \cos A,$$

ce qui donne

$$\Omega\Omega_a \,.\, \Omega\Omega_b = F_1F_1' \,.\, F_2F_2' = OO_a{}^2 \sin^2\theta + LL_a{}^2 \cos^2\theta = R^2 \sin^2\theta.$$

Mais $R^2 \sin^2\theta$ est le carré du demi-axe non focal de l'ellipsoïde (E), ce qui démontre la proposition finale.

7. Tétraèdres cosymédians. Les symédianes AL, BL, CL, DL rencontrent la sphère circonscrite (O, R) aux transformés A_1, B_1, C_1, D_1 des sommets A, B, C, D du tétraèdre T par les inversions (A, λ), (B, λ), (C, λ), (D, λ), λ étant la puissance de L pour la sphère (O, R).

On a donc, d'abord

$$B_1C_1 = a_1 = a \,.\, \lambda/LB \,.\, LC, \quad D_1A_1 = a_1' = a' \,.\, \lambda/LD \,.\, LA, \quad \ldots\ldots\ldots\ldots(4)$$

d'où $\qquad B_1C_1 \,.\, D_1A_1 = a_1a_1' = aa' \,.\, \lambda^2/LA \,.\, LB \,.\, LC \,.\, LD,$

et $\qquad a_1a_1'/aa' = \lambda^2/LA \,.\, LB \,.\, LC \,.\, LD = b_1b_1'/bb' = c_1c_1'/cc'.$

Il en résulte déjà que *les sections antiparallèles aux deux tétraèdres métaharmoniques T et $T_1 \equiv A_1B_1C_1D_1$, dont les côtés sont proportionnels aux produits aa' et a_1a_1', bb' et b_1b_1', cc' et c_1c_1' des arêtes opposées, sont semblables.*

Théorème. *Les tétraèdres T et T_1 ont même angle de Brocard.*

Soient, en effet, θ et θ_1 les angles de Brocard des tétraèdres T et T_1, de volumes V et V_1, tels que

$$\cot\theta = (\Sigma ab'c')/12V, \quad \cot\theta_1 = (\Sigma a_1b_1'c_1')/12V_1.$$

En vertu des relations (4),

$$a_1b_1'c_1' = ab'c' \cdot \lambda^3/(LA \cdot LB \cdot LC)^2, \ldots\ldots,$$

d'où $$\Sigma a_1b_1'c_1' = (\Sigma LA^2) \cdot ab'c' \cdot \lambda^3/(LA \cdot LB \cdot LC \cdot LD)^2.$$

De plus,* $ab'c' = 4R_aS_a = 12\overline{LBCD}/\tan\theta$, $V_1/V = \lambda^4/(LA \cdot LB \cdot LC \cdot LD)^2$; de sorte que

$$(\Sigma LA)^2 \cdot ab'c' = (\Sigma LA)^2 \cdot 12\overline{LBCD} \tan\theta = 12\lambda V,$$

car, d'après un théorème de Faure (*Recueil de théorèmes relatifs aux sections coniques*), $(\Sigma LA)^2 \cdot \overline{LBCD} = \lambda V$.
On a donc finalement,

$$\cot\theta/\cot\theta_1 = \left(\Sigma\frac{ab'c'}{V}\right)\Big/\Sigma\frac{a_1b_1'c_1'}{V_1}\Big) = \lambda V/\lambda V = 1 ;$$

d'où $\cot\theta = \cot\theta_1$, et le théorème est démontré.

Théorème. *Les tétraèdres T et T_1 ont même* second *point de Lemoine L.*
On a, en effet, la relation

$$\overline{A_1B_1C_1} = \frac{\lambda^3}{(LA \cdot LB \cdot LC)^2} \cdot \overline{ABC} \cdot \frac{d_d}{d_{1d}}$$

entre les aires des triangles ABC, $A_1B_1C_1$ et les distances du centre L d'inversion aux plans de ces triangles.
Or, en vertu de (1),

$$d_d = R_d \tan\theta, \quad \overline{ABC} = abc/4R_d, \quad \overline{A_1B_1C_1} = a_1b_1c_1/4R_{1d}.$$

Donc

$$d_{1d} = \frac{\overline{ABC}}{\overline{A_1B_1C_1}} \cdot \frac{\lambda^3}{(LA \cdot LB \cdot LC)^2} \cdot d_d = R_{1d}\tan\theta = R_{1d}\tan\theta_1.$$

Les expressions analogues d_{1a}, d_{1b}, d_{1c} des distances du point L aux plans des faces $B_1C_1D_1$, $C_1D_1A_1$, $D_1A_1B_1$ du tétraèdre T_1 montrent qu'il se confond avec le *second* point de Lemoine de celui-ci.

Corollaire. *Les tétraèdres T et T_1 ont mêmes symédianes AA_1, BB, C_1C_1, DD_1 (tétraèdres cosymédians), même seconde sphère (L) de Lemoine et même ellipsoïde (E) de Brocard.*

Corollaire. *Les sections antiparallèles égales faites dans les trièdres de T et T_1 et passant par L sont des triangles superposables.* Car ces triangles sont semblables entre eux et inscrits dans une même sphère (L).

8. **Tétraèdre de Lemoine.** Les conjugués harmoniques L', L'', L''' du *second* point de Lemoine L du tétraèdre T_1 par rapport aux points où les droites passant par L s'appuient sur les arêtes BC et DA, CA et DB, AB et DC peuvent être appelés *associés harmoniques* de L pour le tétraèdre T. Le tétraèdre $LL'L''L'''$ dont les sommets L', L'', L''' sont dans le plan harmonique ϖ du point L, par rapport à T, est le *tétraèdre de Lemoine.*

Théorème. *Les quatre symédianes du tétraèdre T rencontrent la sphère circonscrite aux sommets d'un tétraèdre $T_1 \equiv A_1B_1C_1D_1$ harmonique au tétraèdre fondamental.*

Les conjugués harmoniques A', B', C', D' des sommets A, B, C, D du tétraèdre T, avec respectivement le point L et le plan $L'L''L'''$ déterminent un tétraèdre $T' \equiv A'B'C'D'$ et les trois tétraèdres T, $LL'L''L'''$, T' forment

* N.B. $\overline{ABC} \equiv$ aire du triangle ABC,
$\overline{ABCD} \equiv$ volume du tetraèdre $ABCD$, etc.

un système desmique. Comme le point L et les pôles du plan $\varpi \equiv L'L''L'''$ par rapport au tétraèdre T et par rapport à la sphère circonscrite à T, se confondent (V. Thébault, *Mathesis*, 1946) le tétraèdre T' est inscrit à celle-ci et ses sommets coïncident avec ceux du tétraèdre cosymédian T_1. Les droites AL, BL, CL, DL étant les symédianes de T, le théorème est démontré.*

Corollaire. *Le tétraèdre $T_2 \equiv LL'L''L'''$ de Lemoine est orthocentrique et la sphère (O, R) est la sphère polaire de T_2. Le centre O de la sphère circonscrite à T est l'orthocentre de T_2.*

Corollaire. *Dans un tétraèdre T, une symédiane est divisée harmoniquement par le second point de Lemoine L et le second point d'intersection de cette symédiane avec la sphère circonscrite.*

En effet, les deux sommets L et D_1 des tétraèdres T_2 et T_1 sont séparés harmoniquement par le sommet D et la face ABC du troisième tétraèdre T du groupe desmique précité.

Corollaire. *Les coordonnées normales des sommets de T_1, par rapport à T, sont proportionnelles à $(-R_a, R_b, R_c, R_d)$, $(R_a, -R_b, R_c, R_d)$, ... et celles des points L', L'', L''' sont proportionnelles à*

$$(-R_a, -R_b, R_c, R_d),\ (R_a, -R_b, -R_c, R_d),\ (R_a, R_b, -R_c, -R_d).$$

9. **Quadriques associées aux tétraèdres cosymédians T et T_1.** D'une manière générale, soient un tétraèdre $T \equiv ABCD$ et un point P de coordonnées normales (x_1, y_1, z_1, t_1) par rapport à T. On considère le réseau ponctuel de quadriques

$$\lambda\left(\frac{yz}{y_1z_1}+\frac{xt}{x_1t_1}\right)+\mu\left(\frac{xz}{x_1z_1}+\frac{yt}{y_1t_1}\right)+\nu\left(\frac{xy}{x_1y_1}+\frac{zt}{z_1t_1}\right)=0.$$

1°. *Toutes les quadriques du réseau passent par les sommets A, B, C, D de T et par les points $A_1(-x_1, y_1, z_1, t_1)$, $B_1(x_1, -y_1, z_1, t_1)$, $C_1(x_1, y_1, -z_1, t_1)$, $D_1(x_1, y_1, z_1, -t_1)$.*

2°. *Le point P a même plan polaire par rapport à toutes les quadriques du réseau.* C'est le plan

$$x/x_1 + y/y_1 + z/z_1 + t/t_1 = 0.$$

3°. *Les tétraèdres T et $T_1 \equiv A_1B_1C_1D_1$ sont polaires réciproques.*

En effet, les droites AP, BP, CP, DP rencontrent les plans des faces BCD, CDA, DAB, ABC de T en A', B', C', D' et les droites A_1P, B_1P, C_1P, D_1P rencontrent les plans $B_1C_1D_1$, $C_1D_1A_1$, $D_1A_1B_1$, $A_1B_1C_1$ en A_1', B_1', C_1', D_1'. La proposition résulte de ce que les divisions (P, A_1, A, A') et (P, A, A_1, A_1'), ... sont harmoniques.

4° *Les tétraèdres T et T_1 sont circonscrits a une même quadrique (Q) qui touche leurs faces en A', B', C', D' et A_1', B_1', C_1', D_1'.*

En effet, la quadrique (Q) d'équation

$$(Q) \equiv \Sigma(x^2/x_1^2) - \Sigma(xy/x_1y_1) = 0,$$

touche le plan BCD sur la droite $APA_1(y/y_1 = z/z_1 = t/t_1)$ et le plan

$$B_1C_1D_1 \equiv -x/x_1 + y/y_1 + z/z_1 + t/t_1 = 0,$$

sur la même droite.

5°. *Les douze arêtes des tétraèdres T et T_1 sont tangentes à une même quadrique (Q') par rapport à laquelle les tétraèdres sont polaires réciproques.*

Car $$(Q') = \Sigma(x^2/x_1^2) - \Sigma(2xy/x_1y_1) = 0.$$

* M. N. Altshiller-Court a signalé ce théorème quand T est *isodynamique* (*Modern Pure Geometry*, p. 283).

Cette quadrique touche les arêtes BC, CD, DB de T aux pieds des céviennes de A' dans le triangle BCD et les arêtes B_1C_1, C_1D_1, D_1B_1 de T_1 aux pieds des céviennes de A_1' dans le triangle $B_1C_1D_1$.*

6°. Dans un tétraèdre T, si le point P se confond avec le *second* point de Lemoine L et la quadrique considérée du réseau avec la sphère circonscrite, toutes ces propriétés s'appliquent aux tétraèdres cosymédians T et T_1.

Théorème. *Deux tétraèdres cosymédians T et T_1 sont circonscrits a un même ellipsoïde (E) aux pieds A', B', C', D' et A_1', B_1', C_1', D_1' de leurs symédianes et leurs douze arêtes sont tangentes à un ellipsoïde (E'), par rapport auquel ils sont polaires réciproques, aux pieds des céviennes de A', B', C', D' et A_1', B_1', C_1', D_1' dans chacune des faces.*

On retrouve ainsi l'ellipsoïde $(Q) \equiv (E)$ de Brocard inscrit aux tétraèdres T et T_1 (P. Delens, *loc. cit.*, Dr. W. L. Marr, *Proc. London Math. Soc.*, 1918-1919, pour T et T_1 *isodynamiques*), et l'ellipsoïde $(Q') \equiv (E')$ que Tarry et Neuberg ont signalé dans le cas particulier où est *isodynamique* (*Association Française pour l'Avancement des Sciences*, 1886, p. 25). V. T.

BIRKBECK COLLEGE.
(UNIVERSITY OF LONDON.)

MATHEMATICAL SEMINAR.

Wednesdays 7-9, *Room* 358 *in the Institute of Education, University of London Buildings, W.C.* 1.

1946.

Oct. 9. *R. Henstock*, Set functions and functional spaces.
Oct. 16. *Dr. M. Loève*, On sets of random events.
Oct. 23. *P. Vermes*, Continuation of Laurent series by infinite matrices.
Oct. 30. *Dr. M. Loève*, Limit processes in the theory of probability.
Nov. 6. *Dr. R. G. Cooke* and *Miss A. M. Barnett*, Some recent work on summability of bounded sequences by infinite matrices.
Nov. 13. *Dr. M. Loève*, Stationary random functions.
Nov. 20. *Dr. R. G. Cooke* and *Miss A. M. Barnett*. Continued from Nov. 6.
Nov. 27. *M. Fréchet*, Professor at the Sorbonne. Abstract spaces in theoretical and applied Statistics.
Dec. 4. *M. Fréchet*. Continued from Nov. 27.
Dec. 11. *H. M. Melvin*. Metric in matrix spaces—an introduction to matrix bounds.

Admission free.

VISUAL AIDS IN MATHEMATICS.

AN exhibition of visual aids in the teaching of mathematics is to be held at the annual general meeting of the Mathematical Association next April. Teachers who have any mathematical models or other suitable material which is available for exhibition are asked to write to Mr. C. T. Daltry at the Institute of Education, Malet Street, London, W.C. 1.

* Ces remarques nous ont été communiquées par M. R. Bouvaist.

MATHEMATICAL NOTES.

1928. *Distribution of curriculum.*

During the war I was able to examine the knowledge and capabilities of the recruits to a pre-service training unit. Ninety of these recruits, of ages between fifteen and seventeen years, had school education only of the type hitherto termed "elementary", such education ceasing between about one and two years before entry to the unit. Although this number is, of course, too small to yield conclusive information, an outstanding consistency in the facts revealed by the investigation (and particularly as regards mathematics) seems to merit attention; and may, indeed, provide a pointer towards the content of "continued education". Especially is this so when it is pointed out that the cadets concerned had, between them, attended some score of different schools, the greatest number from any one school being ten. Further, lest the figures to be given might be regarded as in part attributable to any predilections in my own teaching, it should be added that some of the cadets received their "lessons" from another instructor, results from the several batches being similar. It would appear, therefore, that the possibility of bias of any particular teacher may be ignored.

Upon entry each recruit was put through a routine examination, partly written, partly oral, to determine what portions of the necessary work (i) he had previously been taught, (ii) he still knew. As a result of this his subsequent tuition was arranged, and during the tuition considerable attention was paid to the speed with which a once-possessed facility was restored (although the attention of the pupil was not drawn unduly to this aspect). A tabular summary of the results found in mathematics—"calculations"—brings out the points most clearly. The columns indicate:

(*a*) the number who had been, previously, taught a particular item. (Less reliance can, for obvious reasons, be placed on the figures here than on those in later columns);

(*b*) the number who retained a working knowledge of the item;

(*c*) and (*d*) the number of those, included in (*a*) but not in (*b*), who regained a working knowledge after one lesson and two lessons, respectively.

Item	(*a*)	(*b*)	(*c*)	(*d*)
Tables: Simple Multiplication and Division	90	62	17	4
Money Problems	90	41	26	14
Graphs	70	28	22	8
Averages	80	31	4	8
Manipulation of Fractions	90	19	10	7
Decimals	90	28	11	14
Metric System	62	12	9	13
Percentages	68	20	14	7
Square Root	90	41	26	14

No hard-and-fast deduction is being drawn from this. The possibility of inaccurate responses, the differing desires of various individuals to work, and the existence of border-line cases are not forgotten. Still, it is permissible to suggest (i) that those things learned at the earlier stages are best retained, (ii) that those things with the more practical (and/or visual) aspect are taken to most readily by the adolescent, (iii) that, whatever the reasons, the present system is not yielding over-good products.

Now it may be that the knowledgeable will say that this is nothing new; that it is almost obvious. But is there not a danger, especially in some of

these elementary matters, that the obvious may be overlooked? It is suggested, therefore, more as a basis for discussion than in any dogmatic spirit, that with the extension of the school-leaving age and the creation of " County Colleges " a distinct readjustment of work done can be made. In the primary school more attention, in greater detail, can be paid to the foundations of all subjects ; the new secondary school can consolidate such learning, and the practical applications (*alias* vocational training) can be left to form part of the curriculum of the continuation school. The adolescent will be better able to appreciate the value to his work of the " academic " side of his education ; and he will have a firmer understanding of fundamentals to assist him extend his knowledge and his capacity for life. F. W. K.

1929. *On certain functional equations and their solutions.*

1. If

$$f(z)=e^z, \quad \phi(z)=\sin z, \quad \psi(z)=\cos z, \quad F(z)=\tan z, \quad \ldots\ldots\ldots(1)$$

then

$$\left.\begin{aligned} &f(z_1+z_2)=f(z_1)f(z_2),\\ &\phi(z_1+z_2)=\phi(z_1)\psi(z_2)+\phi(z_2)\psi(z_1),\\ &\psi(z_1+z_2)=\psi(z_1)\psi(z_2)-\phi(z_1)\phi(z_2),\\ &F(z_1+z_2)=\frac{F(z_1)+F(z_2)}{1-F(z_1)F(z_2)}, \end{aligned}\right\} \quad \ldots\ldots\ldots(2)$$

A question that naturally arises is: Given the functional equations (2), are their solutions unique? That is, besides the solutions (1), are there any other solutions of the functional equations (2)? We propose to consider this question by investigating the solutions of (2) which are analytic in domains which include the origin.

2. We take first the equation

$$f(z_1+z_2)=f(z_1)f(z_2). \quad \ldots\ldots\ldots(3)$$

Putting $z_1=z_2=0$, we have $f(0)=f^2(0)$. Hence

$$f(0)=0, \ 1.$$

(*a*) $f(0)=0$ is trivial, since by putting $z_2=0$ in (3), we obtain

$$f(z)\equiv 0.$$

(*b*) Take the case $f(0)=1$. By differentiating (3) partially with respect to z_2 and then putting $z_2=0$, we obtain

$$f'(z_1)=f(z_1)f'(0).$$

Hence integrating with respect to z_1, we have

$$f(z_1)=A\exp\{f'(0)z_1\},$$

where A is a constant which is determined by the condition $f(0)=1$, that is, $A=1$; $f'(0)$ is arbitrary. Hence we have as the complete solution

$$f(z)=0, \quad f(z)=e^{\alpha z}, \quad \ldots\ldots\ldots(4)$$

where α is an arbitrary constant, real or complex.

3. To solve the equation

$$F(z_1+z_2)=\frac{F(z_1)+F(z_2)}{1-F(z_1)F(z_2)}, \quad \ldots\ldots\ldots(5)$$

we note that $$F(0) = 2F(0)/\{1 - F^2(0)\}.$$

Hence $$F(0) = \pm i,\ 0.$$

(*a*) If $F(0) = \pm i$, then putting $z_2 = 0$ in (5), we obtain

$$F(z_1) = \frac{F(z_1) \pm i}{1 \mp iF(z_1)},$$

from which we readily see that $F(z_1) = \pm i$. The verification that this solution satisfies (5) is trivial.

(*b*) If $F(0) = 0$, then by differentiating the relation (5) partially with respect to z_2 and then putting $z_2 = 0$, we have

$$F'(z_1) = [1 + F^2(z_1)]\, F'(0).$$

Hence we obtain on integration

$$\text{arc tan } F(z_1) = F'(0)\,.\,z_1 + C,$$

and the condition $F(0) = 0$ requires that $C = 0$. Therefore

$$F(z) = \tan \alpha z,$$

where $\alpha = F'(0)$ and may be real or complex. Hence the complete solution of (5) is

$$F(z) = \pm i, \quad F(z) = \tan \alpha z. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

4. To solve

$$\phi(z_1 + z_2) = \phi(z_1)\psi(z_2) + \phi(z_2)\psi(z_1), \quad \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

$$\psi(z_1 + z_2) = \psi(z_1)\psi(z_2) - \phi(z_1)\phi(z_2), \quad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

we multiply (7) by i and add to (8). Then if

$$u(z) \equiv \psi(z) + i\phi(z),$$

$u(z)$ satisfies

$$u(z_1 + z_2) = u(z_1)u(z_2).$$

Hence from (3) and (4) we have

$$u(z) = ke^{\alpha z},$$

where $k = 0$ or 1, and $\alpha = u'(0)$ is an arbitrary constant. From (7) and (8), we obtain

$$\phi(z_1 + z_2) = \phi(z_1)\{k \exp(\alpha z_2) - i\phi(z_2)\} + \phi(z_2)\{k \exp(\alpha z_1) - i\phi(z_1)\}. \ldots (9)$$

$\phi(0)$ is given by $$\phi(0) = 2\phi(0)\{k - i\phi(0)\}.$$

Therefore

$$\phi(0) = 0, \quad \tfrac{1}{2}(1 - 2k)i.$$

Putting $z_2 = 0$ in (9) we have

$$\phi(z_1) = \phi(z_1)\{k - i\phi(0)\} + \phi(0)\{k \exp(\alpha z_1) - i\phi(z_1)\}, \quad \ldots\ldots\ldots\ldots (10)$$

and putting $z_2 = 0$ in (9) after differentiating it partially with respect to z_2,

$$\phi'(z_1) = \phi(z_1)\{\alpha k - i\phi'(0)\} + \phi'(0)\{k \exp(\alpha z_1) - i\phi(z_1)\}. \quad \ldots\ldots\ldots\ldots (11)$$

If $k = 0$ and $\phi(0) = 0$, then from (10), $\phi(z) \equiv 0$, and therefore $\psi(z) \equiv 0$.

If $k = 0$ and $\phi(0) = \frac{1}{2}i$, then (10) reduces to an identity, and (11) gives

$$\phi'(z_1) = -2i\phi(z_1)\phi'(0).$$

Therefore integrating with respect to z_1, we obtain

$$\phi(z) = \tfrac{1}{2}i \exp\{-2i\phi'(0)\,.\,z\},$$

and therefore

$$\psi(z) = \tfrac{1}{2} \exp\{-2i\phi'(0)\,.\,z\}.$$

If $k=1$ and $\phi(0)=-\frac{1}{2}i$, then (10) gives $\phi(z)=-\frac{1}{2}ie^{\alpha z}$ and therefore

$$\psi(z)=\tfrac{1}{2}e^{\alpha z}.$$

If $k=1$ and $\phi(0)=0$, then (10) reduces to an identity and (11) gives

$$\phi'(z_1)=\phi(z_1)\{\alpha-2i\phi'(0)\}+\phi'(0)e^{\alpha z_1}.$$

Multiplying by $\exp\{2i\phi'(0)z-\alpha z\}$, and integrating, we have

$$\phi(z)\exp\{2i\phi'(0)z-\alpha z\}=\int\phi'(0)\exp\{2i\phi'(0)z\}\,dz+C.$$

Hence $$\phi(z)=\tfrac{1}{2}i\,e^{\alpha z}[-1+\exp\{-2i\phi'(0)z\}],$$

and $$\psi(z)=\tfrac{1}{2}e^{\alpha z}[1+\exp\{-2i\phi'(0)z\}].$$

These may be written as

$$\phi(z)=e^{\beta z}\sin\gamma z,\quad \psi(z)=e^{\beta z}\cos\gamma z,$$

where $$\beta=\alpha-i\phi'(0)\quad\text{and}\quad\gamma=\phi'(0).$$

Thus the complete solution of the functional equations (7) and (8) is

(i) $$\phi(z)=0,\quad\psi(z)=0,$$

(ii) $$\phi(z)=\pm\tfrac{1}{2}i\,e^{\alpha z},\quad\psi(z)=\tfrac{1}{2}e^{\alpha z},$$

where α is an arbitrary constant,

(iii) $$\phi(z)=e^{\beta z}\sin\gamma z,\quad\psi(z)=e^{\beta z}\cos\gamma z,$$

where β and γ are arbitrary constants. **C. J. E.**

1930. *Addenda to Note* 1845 (*Find the penny*).

The number $\frac{1}{2}(3^n-2n+3)$ of pennies from which a single bad coin may be detected by n weighings, given in Note 1845, is not the best possible, and n weighings suffice for a group of $\frac{1}{2}(3^n-3)$ pennies. The problem has aroused wide interest, and many proofs of the theorem for $\frac{1}{2}(3^n-3)$ pennies, and of variations on the puzzle, have been communicated. Mr. T. H. Wise pointed out that the proof of Note 1845 for $\frac{1}{2}(3^n-2n+3)$ pennies can be applied to the case of $\frac{1}{2}(3^n-3)$ pennies, with the addition of the following Lemma :

Lemma IV. If $C(1, K(n))\neq0$ and $A(1, K(n))=0$, then n weighings suffice to determine the bad penny and its fault.

Mr. Wise gave also an alternative proof, and Mr. R. H. Macmillan proved that n weighings suffice to determine the bad coin (if any) from a set of $\frac{1}{2}(3^n-3)$ coins which *may* contain one bad coin.

Mr. D. E. Littlewood and Pte. G. R. Welic, of the U.S. Army, prove (i) that if a set of $\frac{1}{2}(3^n+1)$ pennies is heavier or lighter than a set of $\frac{1}{2}(3^n-1)$ pennies plus one good penny, then the bad coin may be determined by n weighings ; and (ii) given one good coin and knowing that one bad coin is in a set of $\frac{1}{2}(3^n+1)$ pennies, the bad one can be determined by n weighings.

The most comprehensive discussion of the problem received is that given in the following Note by Mr. F. J. Dyson. The delightful proof of the problem of $\frac{1}{2}(3^n-3)$ pennies by means of " ternary " labels was also discovered by Mr. R. C. Lyness. Other contributors were Mr. H. L. Cox, Mr. F. M. Goldner, Mr. E. Klein, Mr. E. V. Newbery, Mr. W. A. H. Rushton, Dr. A. G. Walker.

1931. *The problem of the pennies.*

The problem is to identify a defective penny among a collection of M pennies of identical appearance, knowing that the defective penny is of a different

weight from the others, by the use of a pair of weighing-scales without weights. In n operations of weighing, it is required to identify the defective penny and to decide whether it is lighter or heavier than the others.

In the *Mathematical Gazette* for December 1945 (Note No. 1845) Mr. R. L. Goodstein gave a solution for the case $M \leqslant \frac{1}{2}(3^n - 2n + 3)$, $n \geqslant 3$. I give here a neater solution for the case $3 \leqslant M \leqslant \frac{1}{2}(3^n - 3)$, and I show that these bounds for M are necessary for the problem to be soluble.

I. *Solution of the problem for* $M = \frac{1}{2}(3^n - 3)$.

Let the pennies be numbered from 1 to M. Each penny is given a "label" consisting of its number expressed as a ternary decimal, with a sufficient number of zeroes placed at the beginning to make the number of digits in each label exactly n. Each penny is then given a second label obtained by subtracting its first label from the ternary decimal $3^n - 1 = 2M + 2$. Each label of a given penny can be derived from the other by changing digits 0 into 2 and digits 2 into 0 while leaving digits 1 unaltered. Every n-digit ternary decimal occurs just once as a label, except for the three consisting of one digit repeated n times.

A label is called "clockwise" if the first change of digit in it after the beginning is a change from 0 to 1 or from 1 to 2 or from 2 to 0; it is called "anticlockwise" if the first change of digit is from 1 to 0 or from 2 to 1 or from 0 to 2. Of the two labels of a given penny, one is clockwise and the other anticlockwise. We denote by $C(i, d)$ the class of pennies whose clockwise labels have d for their i-th digit. A cyclic permutation of digits changing 0 to 1, 1 to 2, 2 to 0 in the labels of all pennies would simply transfer all pennies from $C(i, 0)$ to $C(i, 1)$, from $C(i, 1)$ to $C(i, 2)$, and from $C(i, 2)$ to $C(i, 0)$; this shows that the classes $C(i, 0)$, $C(i, 1)$ and $C(i, 2)$ all contain the same number $\frac{1}{3}M$ of pennies.

The n weighing operations are specified by the rule that at the i-th weighing the pennies of $C(i, 0)$ are weighed in the left-hand pan of the scales against the pennies of $C(i, 2)$ in the right-hand pan, the pennies of $C(i, 1)$ being laid aside. The result of the i-th weighing is symbolised by a number a_i which we take to be 0 if the left-hand pan sinks, 2 if the right-hand pan sinks, and 1 if the scales remain level. We consider the ternary decimal

$$A = 3^{n-1}a_1 + 3^{n-2}a_2 + \ldots + a_n.$$

It follows from the result of the i-th weighing that the defective penny either is heavy and has a_i as the i-th digit of its clockwise label, or is light and has a_i as the i-th digit of its anticlockwise label. Therefore after n weighings the defective penny is determined uniquely as the penny one of whose labels is A, and it is heavier or lighter than the others according as this label is clockwise or anticlockwise. It is interesting to notice that the latter question will usually be decided by the first 2 or 3 weighings.

II. *Solution for* $3 \leqslant M < \frac{1}{2}(3^n - 3)$.

The method of solution for $M = \frac{1}{2}(3^n - 3)$ applies word for word to the case $3 \leqslant M < \frac{1}{2}(3^n - 3)$ if the initial labelling of the pennies is suitably modified. Let the $\frac{1}{2}(3^n - 3)$ possible clockwise labels be divided into groups of three, the members of each group being derived from each other by a cyclic permutation of the digits 0, 1 and 2. The pennies are then assigned clockwise labels in an arbitrary way, except that the labels of each group of three must be used up before another group of three is begun. At the end there may be one or two pennies whose labels belong to an incomplete group of three; if there is one such penny its label must be chosen to be that member of the group whose first digit is 1, and if there are two such pennies their labels must be those members of the group whose first digits are 0 and 2. After all the

pennies have been assigned clockwise labels in this way, anticlockwise labels are added as before by subtracting the first labels from $3^n - 1$.

The weighing can now be carried out exactly as in the case $M = \frac{1}{2}(3^n - 3)$, provided that the sets $C(i, 0)$ and $C(i, 2)$ contain the same number of pennies for a given value of i. This equality is assured by the method of labelling for the first weighing ($i = 1$). At the subsequent weighings there may be a deficiency of one penny at most in either of the sets $C(i, 0)$ and $C(i, 2)$, but this deficiency can always be corrected by adding to the appropriate pan of the scales one of the pennies which has been proved non-defective at the first weighing. The first weighing must always prove some pennies non-defective, provided that none of the sets $C(1, 0)$, $C(1, 1)$ and $C(1, 2)$ is empty, and for this it is necessary and sufficient that $M \geqslant 3$.

III. *Proof of insolubility for* $M > \frac{1}{2}(3^n - 3)$.

Let a solution be possible for M pennies. At the i-th weighing, let $C(i, 0)$ be the set of pennies in the left-hand scale which are weighed against the set $C(i, 2)$ of pennies in the right-hand scale, and let $C(i, 1)$ be the set of pennies not in either scale. The solution must provide a method of choosing the sets $C(i, 0)$, $C(i, 1)$ and $C(i, 2)$ based on a knowledge of the results of the first $(i - 1)$ weighings only. Let $a_i = 0$, 1, or 2 according as the left-hand pan goes down, remains level, or goes up at the i-th weighing. Let D_i be the set of pennies common to all $C(j, a_j)$ for $j \leqslant i$, and E_i the set common to all $C(j, 2 - a_j)$ for $j \leqslant i$. The sum total of knowledge gained from the first i weighings is, that the defective penny is either heavy and in D_i or light and in E_i.

We denote by $N(A)$ the number of pennies in a set A, and by $A \,.\, B$ the set of pennies common to sets A and B. We wish to prove the inequality

$$N(D_i) + N(E_i) \leqslant 3^{n-i}, \quad i = 1, 2, \ldots, n. \qquad \ldots\ldots\ldots\ldots (1)$$

The proof is by reversed induction on i. In the first place, (1) must be true for $i = n$, because the defective penny and its weight relative to the others must be fixed uniquely by the knowledge that it is either heavy and in D_n or light and in E_n. Secondly, assuming that (1) is true for $i + 1$, we can prove it true for i. For before the $(i + 1)$-th weighing it is known that $N(D_{i+1}) + N(E_{i+1})$ will be one of the three numbers :

$$N(D_i \,.\, C(i + 1, a)) + N(E_i \,.\, C(i + 1, 2 - a)), \quad a = 0, 1, 2, \ldots\ldots\ldots (2)$$

but it is not known which. Because the sets $C(i + 1, a)$, $a = 0, 1, 2$, must be specified before the result of the $(i + 1)$-th weighing is known, these sets must be chosen so that (1) is true for $i + 1$ in all three cases. Hence each of the three numbers (2) must not exceed 3^{n-i-1}. But the sum of the three is just $N(D_i) + N(E_i)$, and so (1) is proved for i.

Since (1) is true for $i = 1$,

$$N(C(1, a_1)) + N(C(1, 2 - a_1)) \leqslant 3^{n-1}. \qquad \ldots\ldots\ldots\ldots (3)$$

Now the sets $C(1, 0)$, $C(1, 1)$, $C(1, 2)$ must be specified before the value of a_1 is known, and therefore (3) must be true for each possible value of a_1. Also $N(C(1, 0)) = N(C(1, 2))$, since $C(1, 0)$ and $C(1, 2)$ are weighed against each other. Therefore (3) gives

$$N(C(1, 0)) = N(C(1, 2)) \leqslant \tfrac{1}{2}(3^{n-1}), \quad N(C(1, 1)) \leqslant \tfrac{1}{2}(3^{n-1}).$$

Since all the $N(C(1, a))$ are integers, this gives finally

$$M = N(C(1, 0)) + N(C(1, 1)) + N(C(1, 2)) \leqslant \tfrac{1}{2}(3^n - 3)$$

as a necessary condition for solubility.

IV. *Additional remarks.*

(i) If it is required to determine the defective penny but not whether it is heavier or lighter than the others, the problem is soluble for $M = \frac{1}{2}(3^n - 1)$. The solution is obtained by the method of section I, with one extra penny bearing the label M (*i.e.* a ternary decimal consisting of n 1's).

(ii) If the solver is allowed to take an extra penny which he knows to be of normal weight out of his own pocket, he can solve the problem in its original form with $M = \frac{1}{2}(3^n - 1)$. The solution is obtained by the method of section I, if the $\frac{1}{2}(3^n - 1)$-th penny is given the two labels 0 and $3^n - 1$ (*i.e.* ternary decimals consisting of n 0's and n 2's) fixed by convention as being clockwise and anticlockwise respectively. The extra penny which is known to be normal is not labelled, but is put into the right-hand pan at each weighing in order to balance the scales.

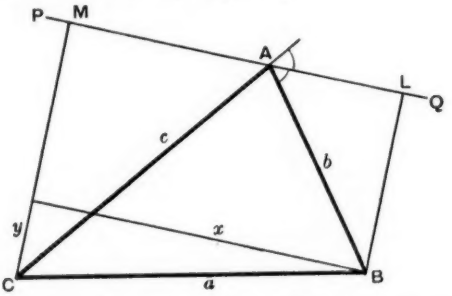
(iii) The problem can be solved for $\frac{1}{2}(3^n + 1)$ pennies if the dispensations (i) and (ii) are both allowed simultaneously. F. J. DYSON.

1932. *Solution of triangles.*

1. If ABC is a plane triangle, in which $b > c$, and PQ is the external bisector of the angle A, then

$$\angle PAC = \angle BAQ = \tfrac{1}{2}(B + C),$$

and the angle between CB and PQ is $\frac{1}{2}(B - C)$.



Let BL, CM be perpendicular to PQ; then, projecting parallel and perpendicular to PQ,

$$a \sin \tfrac{1}{2}(B - C) = CM - BL = (b - c) \sin \tfrac{1}{2}(B + C) = y,$$
$$a \cos \tfrac{1}{2}(B - C) = MA + AL = (b + c) \cos \tfrac{1}{2}(B + C) = x.$$

Thus
$$\tan \tfrac{1}{2}(B - C) = \frac{y}{x} = \frac{b - c}{b + c} \tan \tfrac{1}{2}(B + C),$$

and
$$a = y \operatorname{cosec} \tfrac{1}{2}(B - C) = x \sec \tfrac{1}{2}(B - C) = \sqrt{(x^2 + y^2)}.$$

This argument, which after discovering for myself I found had been given by Mr. J. W. Rogers in Note 1091, *Gazette*, XVII, p. 323, provides a method of solving completely a triangle, given two sides and the included angle, which I think is both new and neat—see examples below. It further leads, with little difficulty, to the " half-angle " formulae, as follows :

$$\begin{aligned} a^2 = x^2 + y^2 &= (b - c)^2 \cos^2 \tfrac{1}{2}A + (b + c)^2 \sin^2 \tfrac{1}{2}A \\ &= (b + c)^2 - 4bc \cos^2 \tfrac{1}{2}A \\ &= (b - c)^2 + 4bc \sin^2 \tfrac{1}{2}A, \end{aligned}$$

whence $\cos^2 \frac{1}{2}A = \{(b+c)^2 - a^2\}/4bc = s(s-a)/bc,$

$\sin^2 \frac{1}{2}A = \{a^2 - (b-c)^2\}/4bc = (s-b)(s-c)/bc,$

and so $\tan^2 \frac{1}{2}A = (s-b)(s-c)/s(s-a).$

The cosine formula can, of course, also be deduced.

2. To solve a triangle completely, given b, c, A.

Using $b-c$, $b+c$ and $\frac{1}{2}(B+C)$ we get x and y (or their logarithms). Hence we get $\frac{1}{2}(B-C)$, which enables us to find not only B and C, but also a, since

$$a = x \sec \tfrac{1}{2}(B-C) \quad (\text{or } y \operatorname{cosec} \tfrac{1}{2}(B-C)).*$$

As a check, either b or c can be recalculated by the sine formula.† It is clearly unnecessary to recalculate both.

3. *Example* 1.

$$a = 316, \quad b = 220, \quad C = 73°, \quad \text{so that} \quad A + B = 107°.$$

		←——logs——→		
$a - b = 96$		1·9823	2·7292	$536 = a + b$
$\frac{1}{2}(A+B) = 53°\ 30'$	(sin)	$\bar{1}$·9052	$\bar{1}$·7744	(cos)
	y	1·8875	2·5036	x
	x	2·5036	0·0124	(sec)
$\frac{1}{2}(A-B) = 13°\ 36'$	(tan)	$\bar{1}$·3839	2·5160	**c = 328**
A = 67° 06'			0·0194	cosec C
B = 39° 54'			$\bar{1}$·9643	sin A
			2·4997	$a = 316{\cdot}0$, check.

Example 2. To find the third side by using squares and square roots (recommended only when the given sides have easy numerical values).

$$b = 8, \quad c = 3, \quad A = 110°, \quad \text{so that} \quad B + C = 70°.$$

	Squares	logs	
$y = 5 \sin 35° = 2{\cdot}868$	8·226	0·4576	
$x = 11 \cos 35° = 9{\cdot}011(2)$	81·20	0·9547	
$\frac{1}{2}(B-C) = 17°\ 40'$	89·22(6)	$\bar{1}$·5029	(tan)

B = 52° 40' **a = 9·456**
C = 17° 20'

The check can now be completed as before.‡ D. F. FERGUSON.

* x sec $\frac{1}{2}(B-C)$ in most cases gives a more accurate result for a than y cosec $\frac{1}{2}(B-C)$.

† Checking the *larger* side in most cases gives a more accurate result.

‡ Further notes to the examples:

(i) All angles used in the *evaluation* are acute.

(ii) If, as can easily happen in the tangent formula, the angles to be found get interchanged, by using $\frac{1}{2}(B-C)$ in cases where it should be $\frac{1}{2}(C-B)$, the error is *at once* shown up by the check.

1933. *A geometrical query.*

In Casey's *Sequel to Euclid*, No. 136 of the Miscellaneous Exercises is to prove that the circumcircle of the diagonal triangle of a quadrangle passes through the point common to the four nine-points circles of the triangles formed by three of the four points of the quadrangle.

It seems, as Coolidge says in his *Circle and Sphere* about a somewhat similar situation, that it is unsporting to bring the circumscribed rectangular hyperbola into a question intended to be done by properties of straight lines and circles. The properties of this figure were discussed in some detail by J. H. Lawlor in Note 516 (*Gazette*, IX, p. 127), but he makes use of the rectangular hyperbola—the only piece of conics in the Note.

As Professor Forder solved Coolidge's problem, perhaps some reader can do the same for this one? H. V. MALLISON.

1934. *A nomogram for the solution of right-angled triangles.*

In the calculations involved in the resolution of forces, the resultants of perpendicular components, operations with complex numbers, etc., it is frequently necessary to convert cartesian to polar coordinates, or *vice versa*, with a moderate degree of accuracy. Traverse tables are helpful, but are bulky and difficult to use in reverse, that is, to find r, θ when x, y are given. There should be (perhaps there is) an instrument for speedy and reasonably accurate operation. An alternative mechanical aid is a nomogram in which corresponding values of all four variables occur in one straight line. Since the values of any two of the variables determine the line, the nomogram can be used to solve six different problems.

The relation $x^2+y^2=r^2$ is easily represented by parallel scales in which each number appears at a distance from the origin proportional to the square of the number. In order that θ also shall be represented, it is sufficient that the line joining the x and y readings shall pass through a point which is fixed if y/x is constant. This is the case; moreover, the locus of the point, as the ratio y/x varies, is the straight line joining the origins of x and y. The most convenient arrangement (Fig. 1) seems to be one in which the scales of x and y are in opposite directions. If the scale of r is $1/k$ times that of y, then the scale of x is $1/(k+1)$ times that of y and $AB:BC=k:1$ where A, B and C are the origins.

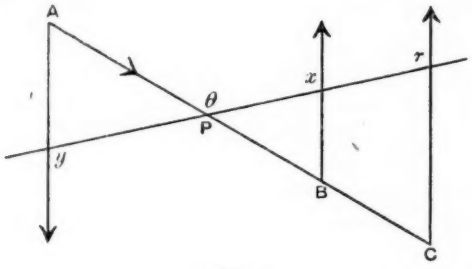


FIG. 1.

Let $z=AP$. Then z represents θ and

$$z \propto \frac{y^2}{y^2+\{x^2/(k+1)\}}=\frac{k+1}{k+1+\cot^2\theta}.$$

The intervals in the scale of θ are at a maximum when $d^2z/d\theta^2$ is zero, and this occurs when $2\cot^2\theta = 3k + \sqrt{(9k^2+4k+4)}$. In the neighbourhood of this maximum the scale is practically uniform. At $\theta = 0$, $d^2z/d\theta^2$ is infinite, so that the scale is bound to be unsatisfactory for very small angles. We must decide the least value of θ for which the nomogram is to be used and choose k accordingly. Choosing $k = 4$, axes of y and r 10 inches apart and 10 inches to represent 10 on the scale of y, a nomogram was constructed which gives very good results for values of θ ranging from 5° to 45°. Its appearance is indicated in Fig. 2.

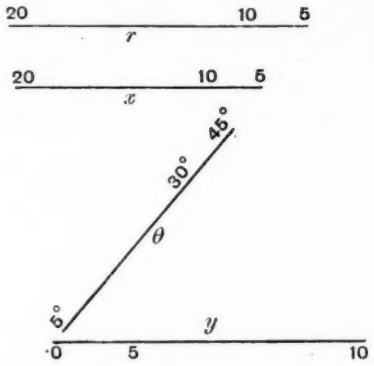


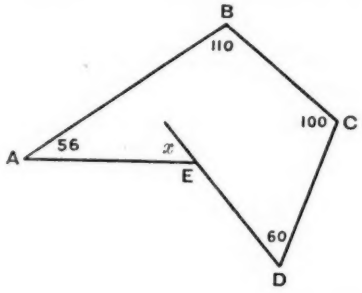
FIG. 2.

The scales of x, y and r are not satisfactory near their origins, but this difficulty is easily overcome by solving a similar triangle. It is worth noting that by multiplying or dividing, if necessary, by 2, the first significant figure of any number can be brought within the range 4 to 10, or within any range such that the ratio of the extremes is 4 : 10. If the smaller acute angle of the triangle which is to be solved lies within the range of values of θ covered by the nomogram, then there must be an easy similar triangle for which the line of section cuts the scales at convenient points. Given good draughtsmanship, average errors in use of the nomogram described should not exceed 3′ in θ and $\frac{1}{4}\%$ in x, y or r. C. G. PARADINE.

1935. *On the angles of a polygon.*

In teaching the elementary properties of polygons, a formula is often given for the sum of the exterior angles when the polygon is convex which presents no difficulty to normal classes. When the similar formula is expounded for re-entrant polygons, however, none but the most intelligent seem capable of appreciating the change of sign or modified definition of exterior angle involved until a much later stage in algebraic technique is reached. Boys of 12–14 can generally work to a rule successfully, so long as no special conditions are attached, so that it seems desirable if possible to shape the rule for both convex and re-entrant polygons simultaneously. This is not difficult, but I have not found it in any of my textbooks. To the young mind the " exterior " angle of a convex or re-entrant polygon means the whole angle, but " exterior " for geometrical purposes unfortunately has a special meaning coupled with the supplementary angle. To avoid this snag, suppose we call the whole external angle the " *outer* " angle. Then the

sign difficulty in the usual application of the formula is automatically avoided. We state that the sum of the " outer " angles of any polygon, whether convex or re-entrant, is equal to $360° + n \,.\, 180°$, where n is the number of angles. This is as easy to prove for a convex polygon as the usual method, and since the proof anyway is glossed over for re-entrants, we have given the practical rule without the need for any academic subtlety. Consider as an explanatory example the figure below.



To find $\angle AED$.

Method 1. Calculate the exterior angles at A, B, C, D and call the interior supplement of $\angle AED$, $x°$. (This is a big snag.) Then

$$124° + 70° + 80° + 120° - x° = 360°.$$

Thus $$x° = 394° - 360° = 34°,$$

and $$\angle AED = 146°.$$

Method 2.

$$304° + 250° + 260° + 300° + \angle AED = 360° + 5 \,.\, 180°$$
$$= 1260°.$$

Hence $$\angle AED = 1260° = 1114° = 146°.$$

(We use bigger numbers, but the result is assured.)

Once the idea of the " outer " angle is absorbed, as distinct from the " exterior " angle, a whole range of simple calculations previously beyond young people can be entered into safely. Of course, it is obvious that, if preferred, the rule may be stated as: "Take 180° away from each 'outer' angle, add the results to get 360°." Thus

$$(\angle AED - 180°) + (\text{sum of other " exterior " angles}) = 360°.$$

J. I. MASON.

INSTITUTO ROMANO DI CULTURA MATEMATICA.

THE Institute of Mathematical Culture, founded in Rome three years ago, meets each week for a lecture or discussion on the teaching of mathematics, particularly at the secondary school level. The Institute is anxious to have an account of English views on this topic, and any member of the Association visiting Rome who would assist the Institute in this way is invited to inform Dr. Emma Castelnuovo, Via Boncompagni 16, Rome.

The New York Mathematical Tables Project. IV.

Tables of Associated Legendre Functions. Pp. xliv, 306. $7\frac{1}{4}'' \times 10\frac{1}{2}''$. 1945. $5.00; 30s. Prepared by the Mathematical Tables Project, conducted under the sponsorship of the National Bureau of Standards; technical director, Arnold N. Lowan. (Columbia University Press, New York. British Agents, Scientific Computing Service, 23 Bedford Square, London, W.C. 1.)

This volume (the twelfth in the series to be reviewed, see *Math. Gazette*, **29**, 1945, pp. 29–33, 86–87, **30**, 1946, pp. 49–52) contains a collection of tables of Associated Legendre Functions. The preface states that these were prepared "to meet urgent needs for a table to about six significant figures at intervals of 0·1. At this interval of the argument, interpolation in these functions is not everywhere satisfactory; a very considerable amount of further work would be required, in order to make all these functions interpolable over the entire range covered.

"The subtabulation of these functions is on the agenda of the Mathematical Tables Project, but the amount of time which may be devoted to this task and therefore the eventual date of completion is necessarily uncertain. For this reason it was decided to publish all material available at present; any further tabular material relating to these functions which may be completed subsequently will appear in a supplementary volume."

The main tables give values as follows:*

1. $P_n^m(\cos\theta) = \sin^m\theta \dfrac{d^m P_n(\cos\theta)}{d(\cos\theta)^m}$; 2. $\dfrac{d}{d\theta} P_n^m(\cos\theta)$;

$n = 1(1)10$, $m = 0(1)4$, $m \leqslant n$; $\theta = 0°(1°)90°$, 6 figures.

3. $P_n^m(x) = (x^2-1)^{\frac{1}{2}m} \dfrac{d^m P_n(x)}{dx^m}$; 4. $\dfrac{d}{dx} P_n^m(x)$;

5. $Q_n^m(x) = (x^2-1)^{\frac{1}{2}m} \dfrac{d^m Q_n(x)}{dx^m}$; 6. $\dfrac{d}{dx} Q_n^m(x)$;

$n = 1(1)10$, $m = 0(1)4$, $m \leqslant n$; $x = 1(0{\cdot}1)10$, 6 figures or exact.

7. $P_n^m(ix)$; 8. $\dfrac{d}{dx} P_n^m(ix)$; 9. $Q_n^m(ix)$; 10. $\dfrac{d}{dx} Q_n^m(ix)$;

$n = 0$ or $1(1)10$, $m = 0(1)4$, $m \leqslant n$; $x = 0(0{\cdot}1)10$, 6 figures or exact.

11. $P^m_{n+\frac{1}{2}}(x)$; 12. $\dfrac{d}{dx} P^m_{n+\frac{1}{2}}(x)$; 13. $Q^m_{n+\frac{1}{2}}(x)$; 14. $\dfrac{d}{dx} Q^n_{m+\frac{1}{2}}(x)$;

$n = -1(1)+4$, $m = 0(1)4$; $x = 1(0{\cdot}1)10$, mainly 6 figures.

Supplementary tables give:

15. $P_n^m(x), \dfrac{d}{dx} P_n^m(x)$; $n = 4$ or $5(1)10$, $m = 0, 2, 4$;

* In these descriptions $P_n(x)$ is the Legendre Polynomial of degree n, when n is an integer, and $Q_n(x)$ is the corresponding Legendre function of the second kind. We use the NYMTP notation $P_n^m(x)$, etc., for the associated functions, rather than $P_{nm}(x)$, etc., as adopted in the *Index of Mathematical Tables*, by A. Fletcher, J. C. P. Miller and L. Rosenhead, 1946.

16. $P_n^m(x)/(x^2-1)^{\frac{1}{2}m}$, $\left[\frac{d}{dx}P_n^m(x)\right]\Big/(x^2-1)^{\frac{1}{2}m-1}$; $n=2$ or $3(1)10$, $m=1,3$;

both tables for $x=1(\cdot1)2$, exact values, up to 24 figures.

17. $(x^2-1)^{\frac{1}{4}}\,.\,P^m_{n+\frac{1}{2}}(x)$, $(x^2-1)^{\frac{1}{4}}\frac{d}{dx}P^m_{n+\frac{1}{2}}(x)$; $n=-1(1)+4$, $m=1,3$;

$x=1(\cdot1)2$, 4 to 7 decimals, up to 9 figures.

18. The Legendre Normalizing Factor $N_n^m=\sqrt{\frac{2}{2n+1}\cdot\frac{(n+m)!}{(n-m)!}}$;

$m=0(1)10$, $n=m(1)10$; 10 figures.

19. The First Eleven Coefficients in the Expansion of

$$CF(m+\tfrac{1}{2},\tfrac{1}{2}-m;\ n+\tfrac{3}{2};\ -t);\quad n=1(1)10,\ m=0(1)4;$$

first coefficient to 10 decimals, with exact values of the ratios of successive subsequent coefficients.

The entries in the main tables are all given in the form $A\times10^p$, A and p being tabulated.

One omission may perhaps be pointed out, not as a criticism but merely to indicate a gap that the extent of the present work has made evident, and as a suggestion for the promised supplementary volume. Although tables of $Q_n(x)$ exist (see *Index of Mathematical Tables*, Art. 16·7, p. 241), there seems to be no table of associated Legendre functions of the second kind, $Q_n^m(x)$, for $0\leqslant x\leqslant1$.

As mentioned above the tables are not always interpolable in a satisfactory manner. For each of the main tables, however, there is given a two-page schedule indicating the accuracy attainable by interpolating by means of Lagrangian formulae using from 2 points (linear interpolation) to 6 points.

The volume has an introduction that merits close study; it contains an account of the methods of computation and checking used, and gives a number of new and interesting formulae. A Bibliography lists 47 items containing tabular material connected with Legendre and related functions.

It seems of interest to give the connections between some of the Legendre Functions of half-integral order and the complete Elliptic Integrals;* these could have provided independent checks, noted on p. xxiii of the Introduction as lacking. The existence of such relations, pointed out in the *Index of Mathematical Tables*, p. 238 (the identification and method are due to A. Fletcher), does not seem to be widely realised. The elliptic integrals used below are functions of the moduli k, k', k_1, k_1' given by

$$k^2=\frac{2}{x+1},\ k'^2=\frac{x-1}{x+1},\ k_1^2=\frac{1}{z^2},\ k_1'^2=\frac{z^2-1}{z^2},$$

in which $z=x+\sqrt{x^2-1}$, so that $x=(z^2+1)/2z$ and $\sqrt{x^2-1}=(z^2-1)/2z$; note that $k_1=x-\sqrt{x^2-1}=(1-k')/(1+k')$, corresponding to a Landen transformation. The complete elliptic integrals of the first and second kinds for these moduli are denoted respectively by K, E, K', E', K_1, E_1, K_1', E_1'.

The relations were all derived from the representations in terms of hypergeometric functions given in E. W. Hobson's *The Theory of Spherical and Ellipsoidal Harmonics*, 1931, pp. 61, 201, 203, 210, 235, 304, by means of the recurrence relations for the hypergeometric function and the special forms

$$F(\tfrac{1}{2},\tfrac{1}{2};\ 1;\ k^2)=K/\tfrac{1}{2}\pi$$
$$F(-\tfrac{1}{2},\tfrac{1}{2};\ 1;\ k^2)=E/\tfrac{1}{2}\pi$$

and are as follows, for $x>1$:

$$P_{-\frac{1}{2}}(x)=\frac{2}{\pi}\sqrt{\frac{2}{x+1}}\,K'=\frac{2}{\pi}kK'=\frac{2}{\pi\sqrt{z}}K_1',$$

$$P_{\frac{1}{2}}(x)=\frac{2}{\pi}\sqrt{\frac{2}{x+1}}\{(x+1)E'-K'\}=\frac{2}{\pi}\sqrt{z}E_1',$$

$$P_{\frac{3}{2}}(x)=\frac{2}{3\pi}\sqrt{\frac{2}{x+1}}\{4x(x+1)E'-(4x+1)K'\}$$

$$=\frac{2}{3\pi\sqrt{z}}\{2(z^2+1)E_1'-K_1'\}.$$

$$Q_{-\frac{1}{2}}(x)=\sqrt{\frac{2}{x+1}}\,K=kK=\frac{2}{\sqrt{z}}K_1,$$

$$Q_{\frac{1}{2}}(x)=\sqrt{\frac{2}{x+1}}\{xK-(x+1)E\}=2\sqrt{z}\,.\,(K_1-E_1),$$

$$Q_{\frac{3}{2}}(x)=\frac{1}{3}\sqrt{\frac{2}{x+1}}\{(4x^2-1)K-4x(x+1)E\}$$

$$=\frac{2}{3\sqrt{z}}\{(2z^2+1)K_1-2(z^2+1)E_1\}.$$

$$P^1_{-\frac{1}{2}}(x)=-\frac{1}{\pi}\sqrt{\frac{2}{x-1}}\,(K'-E')=-\frac{1}{\pi\sqrt{z}}\frac{(z^2+1)K_1'-2z^2E_1'}{z^2-1},$$

$$P^1_{\frac{1}{2}}(x)=+\frac{1}{\pi}\sqrt{\frac{2}{x-1}}\,(xE'-K')=+\frac{1}{\pi\sqrt{z}}\frac{(z^2+1)E_1'-2K_1'}{z^2-1},$$

$$Q^1_{-\frac{1}{2}}(x)=-\frac{E}{\sqrt{2(x-1)}}=-\frac{1}{\sqrt{z}}\left\{\frac{2z^2}{z^2-1}E_1-K_1\right\},$$

$$Q^1_{\frac{1}{2}}(x)=-\frac{xE-(x-1)K}{\sqrt{2(x-1)}}=-\sqrt{z}\,.\left\{\frac{z^2+1}{z^2-1}E_1-K_1\right\}.$$

All these relations have been verified numerically with $x=3{\cdot}5$.

For the sake of greater completeness, it seems worth while to give also identifications for $|x|<1$, although not immediately relevant to the volume under review ; the moduli are $k=\sqrt{\frac{1}{2}(1-x)}$, $k'=\sqrt{\frac{1}{2}(1+x)}$.

$$P_{-\frac{1}{2}}(x)=\frac{2}{\pi}K. \qquad Q_{-\frac{1}{2}}(x)=K'\mp iK.$$

$$P_{\frac{1}{2}}(x)=\frac{2}{\pi}(2E-K). \qquad Q_{\frac{1}{2}}(x)=2E'-K'\pm i(2E-K).$$

The double values for $Q_n(x)$ occur because the straight line joining $\zeta=-1$ and $\zeta=+1$ is taken as a cut in the plane of the complex argument $\zeta=x+iy$ of $Q_n(\zeta)$ as usually defined ; the real part, or mean, of these values provides a satisfactory second solution of Legendre's equation for half-integral order when $|x|<1$.

Almost all of the tables in this volume break entirely new ground. H. Tallqvist, *Acta. Soc. Sci. Fenn.*, 33 (9), 1906, gave $P_n^m(\cos\theta)$ for $n=1(1)8$, $m=1(1)n$; $\theta=1°(1°)90°$ to about 10 figures ; this seems to be the only sub-

stantial overlap of a published table with the work now reviewed.* Although Tallqvist's table was known to the NYMTP, their Table I seems to be the result of a fresh computation (this may seem surprising in view of the remarks on the time available for the production of a supplementary volume, but is almost certainly a legacy from the pre-war period, when a main task of the organisation was to provide employment on a substantial scale); the duplication provides a most valuable check on the values, and it is useful to have the table in a more accessible form than previously. Apart from this table, the NYMTP volume is a substantial and important addition to the list of pioneer tables.

Reviews of earlier volumes have indicated that there is no need to test the accuracy of the tables. Nevertheless, an opportunity for comparison with the B.A. Tables, Part-Volume A, reviewed below, was taken and 640 values in common were compared without revealing any discrepancy. Some comparison of Table I with Tallqvist's tables (about 1600 values) was also made with the same result; it is probable that NYMTP workers also made this comparison, although this does not seem to be stated.

The tables have been well-reproduced by photographic methods from typescript, and have a good appearance.

This volume of tables is undoubtedly a very valuable addition to the NYMTP series and, as remarked above, to the list of pioneer tables, in view of its wealth of new material. It forms an excellent and convenient collection of tables of Legendre and related functions. J. C. P. MILLER.

BRITISH ASSOCIATION TABLES.

1. British Association for the Advancement of Science. Mathematical Tables, Part-Volume A. **Legendre Polynomials.** Pp. 42. 8s. 6d. 1946.

2. B.A.A.S. Mathematical Tables, Part-Volume B. **The Airy Integral.** Prepared by J. C. P. MILLER. Pp. 56. 10s. (including Auxiliary Tables I and II). 1946.

3. B.A.A.S. Auxiliary Tables, Number I. **Coefficients in the Modified Everett Interpolation Formula.** 6d. each or 5s. per dozen, postage extra. 1946.

4. B.A.A.S. Auxiliary Tables, Number II. **Table for Interpolation with Reduced Derivatives.** 6d. each or 5s. per dozen, postage extra. 1946.

(All four tables have been prepared by or for the Committee for the Calculation of Mathematical Tables, and published for the British Association at the University Press, Cambridge.)

From 1931 onwards, nine volumes of British Association Mathematical Tables have been published, and through the excellence of both tabulators and printers have established themselves as a series of international repute. In order that the publication of tables too small to form volumes may not be delayed, the Committee has now "decided to begin the publication of a new series of tables, in the form of Part-Volumes. . . . It is intended that, in course of time, these Part-Volumes, or such of them as are deemed suitable, shall be combined into volumes and take their place in the Association's main series of tables." If we foresee tribulation for librarians and bibliographers,

* Tables of $P_n(x)$ for $1 \leqslant x \leqslant 11$, prepared by L. J. Comrie (see the Report of the Mathematical Tables Committee, *B.A. Report* for 1932, pp. 263–264), of $P_n^m(ix)$ and $Q_n^m(ix)$ by J. N. Sarmousakis (see *Math. Tables and other Aids to Computation*, 1, 127, 1943) and several tables collected or compiled by H. T. Davis (see p. xlvi of the book under review) have, however, existed in manuscript for some years; the B.A. tables have now been published (see the review below).

we can only offer our sympathy and proceed to welcome the new publications. They amply maintain the standard of the original series, to which they conform in page size and typography.

The two Auxiliary Tables are each printed on a card rather smaller than the quarto page of the Mathematical Tables. A copy of each card is included in Part-Volume B, but prices are given for the information of those who need separate copies of the cards.

We now consider each publication in turn.

(1) Part-Volume A contains tables of the ordinary Legendre polynomials $P_n(x)$, designed and prepared by Dr. L. J. Comrie, who also contributes a short introduction.

The tables for $n = 2(1)12$, $x = 0(\cdot 01)1$ occupy 6 pages. The values, when not exact, are given to 7 decimals, with second or second and fourth differences, modified when necessary. For $n = 5(1)8$, use has been made of existing tables by the British Association, Tallqvist and Hayashi; values for $n = 10(1)12$ were computed by Dr. A. J. Thompson.

The tables for $x > 1$ occupy 31 pages, and are much more extensive than existing tables of $P_n(x)$ for $x > 1$. They are for $n = 2(1)12$, $x = 1(\cdot 01)6$, with an extension for $n = 2(1)6$, $x = 6(\cdot 1)11$. It is stated that 7 or 8 figures are given "throughout", but nine 6-figure values occur at the top of page A32. Second or second and fourth differences, modified when necessary, are again given.

The reputation of the designer of the tables, as well as that of the distinguished author of *Logarithmetica Britannica*, guarantees the accuracy of the calculations.

Page A5 contains a useful list of formulae, although the condition for the validity of the expansion of the generating function $1/\sqrt{(1 - 2xz + z^2)}$ is hardly $z < 1$.

(2) Part-Volume B is devoted to tables of solutions of the differential equation $y'' = xy$. The solutions may be expressed in terms of Bessel functions of orders $\pm\frac{1}{3}$, but this representation has not been found convenient and no use has been made of it in the calculations. Instead, the analysis has been developed independently along lines initiated by Dr. Harold Jeffreys. Two independent solutions, $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$, which tend to 0 and $+\infty$ respectively as x tends to $+\infty$, are defined by means of definite integrals. The integral for $\mathrm{Ai}(x)$ is

$$\frac{1}{\pi}\int_0^\infty \cos\left(\tfrac{1}{3}t^3 + xt\right)dt,$$

which, apart from changes of scale, is identical with the well-known integral

$$\int_0^\infty \cos\tfrac{1}{2}\pi(w^3 - mw)\,dw,$$

encountered by Airy, the seventh Astronomer Royal, in an optical context in 1838. For this reason $\mathrm{Ai}(x)$ is called the Airy integral. Power series (convergent for all values of x) and asymptotic expansions for large x are obtained for both functions.

Dr. J. C. P. Miller's 13-page introduction covers much ground, and § 4 in particular is of general mathematical interest. The comments in § 5·2 on direction of numerical integration are also worthy of special notice. Full consideration is given to methods of interpolation, especially in connection with the use of Auxiliary Tables I and II. An excellent graph on page B16 is followed by a synopsis of formulae on page B17. Anyone who wishes to test his analytical powers may be recommended to try his hand on some of the results given in the introduction and synopsis.

Except that formulae relating to auxiliary functions and zeros occupy page B48, pages B18–B56 contain the tables, computed under the leadership of Dr. Miller. In general, they are to 8 figures. $\mathrm{Ai}(x)$ and $\mathrm{Ai}'(x)$ are tabulated for $x = -20(\cdot01)+2$, $\log_{10}\mathrm{Ai}(x)$ and $\mathrm{Ai}'(x)/\mathrm{Ai}(x)$ for $x = 0(\cdot1)\,25(1)\,75$, $\mathrm{Bi}(x)$ and "reduced derivatives" for $x = -10(\cdot1)+2\cdot5$, and $\log_{10}\mathrm{Bi}(x)$ and $\mathrm{Bi}'(x)/\mathrm{Bi}(x)$ for $x = 0(\cdot1)\,10$. The first 50 zeros and turning values of $\mathrm{Ai}(x)$ and $\mathrm{Ai}'(x)$, and the first 20 zeros and turning values of $\mathrm{Bi}(x)$ and $\mathrm{Bi}'(x)$, are also given. Moreover, auxiliary functions $F(x)$, $\chi(x)$, $G(x)$, $\psi(x)$ defined by

$$\mathrm{Ai}(x) = F(x)\sin\chi(x), \qquad \mathrm{Bi}(x) = F(x)\cos\chi(x),$$
$$\mathrm{Ai}'(x) = G(x)\sin\psi(x), \qquad \mathrm{Bi}'(x) = G(x)\cos\psi(x),$$

are tabulated for $x = -80(1)-30(\cdot1)+2\cdot5$. Sufficient even differences (modified when necessary) or "reduced derivatives" are given throughout.

It is difficult to imagine that many users of the functions will require more than is here provided. Rarely can the production of a modest 56 pages have cost more effort. Dr. Miller and his various theoretical and computational helpers may be warmly congratulated on the successful outcome of their labours.

(3) Auxiliary Table Number I is printed on one side of the card and gives coefficients for Everett interpolation with modified even differences. The coefficients are given at interval ·01. The ordinary Everett coefficients of the second differences, exact in 7 decimals, have their second differences added, for use in interpolating these interpolation coefficients.

When second differences in a table have been modified, most of the effect of the fourth differences is taken into account in second-difference interpolation. When this does not suffice, the fourth differences remain to be multiplied by only residual coefficients. These are provided in the table. Formulae at the foot show how, by a powerful simultaneous modification of second and fourth differences, the table-maker can enable the table-user to take most of the effect of higher differences into account, while performing only fourth-difference interpolation.

(4) Auxiliary Table Number II is printed on both sides of the card. At interval ·01 in θ, one side tabulates exact values of θ^2, θ^3 and θ^4 and abbreviated values of θ^5, θ^6 and θ^7, for use in interpolation by the Taylor formula

$$f(x+\theta h) = f(x) + \theta\tau + \theta^2\tau^2 + \ldots + \theta^n\tau^n + \ldots,$$

where $\tau^n \equiv \tau^n f(x) = h^n f^{(n)}(x)/n!$, h being the interval of the table of $f(x)$. τ^n is called the nth "reduced derivative" of $f(x)$; some would call it the nth variation. The other side tabulates exact values of 2θ, $3\theta^2$ and $4\theta^3$ and abbreviated values of $5\theta^4$, $6\theta^5$, $7\theta^6$ and $8\theta^7$, for use in finding the first derivative from

$$hf'(x+\theta h) = \tau + 2\theta\tau^2 + 3\theta^2\tau^3 + \ldots + n\theta^{n-1}\tau^n + \ldots.$$

The factor h multiplying $f'(x+\theta h)$ is unfortunately omitted on the card, but is correctly inserted in Part-Volume B. [The error is corrected except on cards sold before its discovery by the reviewer.]

Both cards are certain to be found very useful by computers.

A. FLETCHER.

School Arithmetic. By W. P. WORKMAN. 3rd Edition, revised by G. H. R. NEWTH. Pp. viii, 548. 5d. 6d. (5s. without Answers). 1946. (University Tutorial Press)

This book is a revised edition of an arithmetic textbook which has been widely known and used for nearly half a century. In the present revision, the opportunity has been taken to reduce or omit topics not now included in School Certificate examination schedules, and to bring up to date the subject-

matter of the examples, but in other respects the book remains substantially the same as in its original edition. It is claimed that the book covers the syllabuses of the various School Certificate examinations.

The arrangement of the book is to deal with each of the usual topics of arithmetic separately and in succession. The book opens with a Section revising the Four Rules as applied to a number; the following Sections treat in turn Weights and Measures, Factors and Multiples, Vulgar and Decimal Fractions, the Metric System, Averages, Practice, and Ratio and Proportion. The main applications of Arithmetic are next given in Section VIII, which covers 115 pages and includes Percentages, Interest Simple and Compound, Stocks, Exchanges, Areas and Volumes. There follows a Section on Problems concerning Work, Supply Pipes, Clocks, etc., and the course concludes with a Section on Square Root, Logarithms and Graphs. In consequence of this method of arrangement, the textbook is not one that many teachers would wish to work straight through. For instance, the Unitary Method does not occur until page 210, areas of rectangles are not treated until page 320, and logarithms do not become available until the end of the book. On the other hand, the subject-matter is clearly arranged, and it is easy to find quickly a set of examples on any desired topic.

Plenty of examples are provided; there is a sufficiency of straightforward exercises, which appear to be well-graded, as well as more difficult exercises and problems. A set of 500 Miscellaneous Examples and 160 Examination Questions are given at the end of the text. Explanations of methods are given throughout the book, and there is a good supply of worked examples. The book is clearly printed and the type is large and distinct. Tables of Logarithms, Antilogarithms, and Prime Numbers (1–8999) are included.

C. T. L. C.

Éléments de calcul infinitésimal. Par A. GROSREY. Pp. 192. 280 fr. 1945. (Gauthier-Villars)

This is a concise text, written in the lucid style which one associates with French authors. Its sub-title, "Ouvrage destiné aux étudiants des écoles techniques et adopté par le Département de l'Instruction publique du canton de Genève, pour les élèves du Technicum", indicates its purpose and scope.

Starting from the notion of limits and derivatives, it covers the usual ground of differentiation and integration, infinite series, applications of integration to geometry and mechanics, centres of gravity, centres of pressure, moments of inertia, etc., complex numbers, elementary ordinary differential equations, Fourier series, curvilinear integrals and approximate methods for solving equations.

As might be imagined, to cover this in 174 pages of text necessitates a very condensed treatment, so that the course is essentially in the nature of an outline, setting out the principles of the subject. Nevertheless, these principles are clearly explained, and the thread runs clearly throughout. Further, the exercises at the end of each chapter are good and sufficient in number.

The English reader, familiar with our own school books on calculus, will not fail to be struck by the fact that the standard methods of evaluating indefinite integrals are dismissed in five pages, though a list of 64 integrals is given at the end of the book. Though many of us as teachers believe that too much time is normally spent on evaluating difficult integrals and summing complicated series, which are properly matters for tabulation by computers, nevertheless it is vital that the student should become competent to manipulate the substitutions and other processes which would reduce an integral to a form likely to be found in the tables, and this does necessitate a fair amount of practice and guidance.

It is to be regretted that in the section on moments of inertia there is no mention of the theorem of parallel axes, although the theorem of perpendicular axes is given. It is also a pity that no use is made of hyperbolic functions, which are mentioned, as names only, on p. 143 (*after* integration), but are then dropped immediately. There are, unfortunately, a number of misprints, some of which may confuse the learner, such as (in the same line on p. 102) $\sin\theta = \frac{1}{2}(1 - \cos 2\theta)$, $\cos\theta = \frac{1}{2}(1 + \cos 2\theta)$ and three wrong signs in the table of integrals. But these will probably disappear in a second impression.

The printing is good and the general lay-out gives the book a pleasant and friendly appearance. H. A. H.

Modern mathematics for T. C. MITS (The Celebrated Man In The Street). By H. G. and L. R. LIEBER. Pp. 230. 7s. 6d. 1946. (Allen & Unwin)

This strange book contains many surprising things, but none more surprising than the statement that " it is produced in complete conformity with the authorised [U.S.A.] econ*o*my standards ".

The print is large and good, and if ever there was a book which lived up to Sir Benjamin Backbite's ideal—" a neat rivulet of text shall meander through a meadow of margin "—this is the book.

A couple of specimens will make this clear. In printing them, and later quotations, *each semi-colon denotes the end of a line.* Here is the official explanation of the arrangement : " This is not intended to be ; free verse ; Writing each phrase on a separate line ; facilitates rapid reading ; and everyone; is in a hurry ; nowadays ; " [It might perhaps also be suggested that doubling or trebling the amount of paper used prevents the book appearing too slim for the 7s. 6d. charged.] Here is a passage very characteristic of the latter part of the book : " In other words ; one way in which ; the mathematician is enabled ; to make up a new system ; is ; to take some ; old familiar word ; like ' parallel ' ; examine into its various properties ; retain some of them but ; discard others ; thus obtaining ; a new freedom ; without entirely ; cutting loose from the past ; ".

The book is lavishly illustrated. In addition to many diagrams, it contains 44 full-page line-drawings. These are amusing and attractive and show considerable quality in the artist, and without undue effort it is possible in the case of about one in three to discover some connection with the text, as, for example, when a picture of a man and woman reclining on a couch illustrates the wedding of algebra and geometry due to Descartes. The last eight pictures are to illustrate the new freedom of art (to match the new freedom of the types of mathematics in which $2 + 2 \neq 4$ and $a \,.\, b \neq b \,.\, a$), and are in the style of Picasso though less repulsive, because first they have no dreadful colouring, and secondly they aim at being graceful if unintelligible ; there is, for example, quite an attractive pyramid of eight faces, growing each out of the one below and getting smaller as they go up. Some of the others provide interesting puzzles as to which way up it is best to look at them.

Leaving the illustrations and turning to the text, it should be understood that the aim is not to teach mathematics to those who know none, nor yet to remind those who have at one time learned some mathematics but have forgotten it, but rather to draw lessons for ordinary life from the ideas and methods of mathematics. No doubt some " reminding " is done as well.

The book is divided into two parts, the old and the new.

From the old mathematics T. C. MITS should learn that he must reason carefully and " check his hunches "—this being proved by three excellent, and well-known, examples of cases where a wrong conclusion is natural and

almost inevitable to the unlearned—one of these is usually associated with the chess-board. He must also learn from the history of "useful" inventions which when traced back are found to be based on "useless" mathematical ideas, and to have involved the efforts of men of many nations that "cooperation is essential; for accomplishing things; ... all are needed; to do the job; is not this DEMOCRACY?" An account of the "wedding" already mentioned and its "offspring" calculus is given in more detail—more reminding is perhaps needed here—and MITS learns that calculus is "an instrument; for the study of; a swift dynamic world;".

Last, the ideas of Part I are restated with the various morals to be drawn from them.

Part II deals with mathematics invented since Newton's time and not taught at all to the average man at present.

A well-known proof that every triangle is isosceles, obtained by ignoring the distinction between the inside and outside of a triangle, leads to the moral, "Be reasonable; by bringing to light; your tacit ideas;". Next the history of the invention of non-euclidean geometry, with two parallels to a given line through a point, and the discovery that this applies on a pseudosphere, leads to the idea that self-evident truths should be replaced by assumptions, as to which the authoress says, "I find that; in some ways; I have a good deal of freedom; and in other ways; I am bound tight; thus I am; free to select any; basic assumptions I please; except; that they; MUST NOT CONTRADICT EACH OTHER;".

Next, to "limber up your mind; so beautifully that; you will be prepared to; glide through this changing world; with ease;" comes an explanation of a finite geometry with only 25 points. A triangle and congruence have curious new meanings but many Euclidean theorems, *e.g.* the concurrence of the medians of a triangle at a point of trisection of each, are still true. This is a very attractive chapter.

Next after the "parallel axiom" come other basic truths to be upset. Thus the chapter "Twice two is not four" explains that for two forces of two pounds each at right angles the chapter-heading is justified, and proceeds to give an example of a finite algebra, with only nine numbers in it (whose "sum" or "product" can be found from the "sum table" and "product table" given), which is asserted to have as its basic postulates "nearly all the postulates of; ordinary Algebra; except one;" thus giving us the idea that "various Algebras; like various Geometries; are possible; that twice two; may be four or not four; depending upon; the Algebra in question; that all these; Algebras and Geometries are; MAN-MADE; that, therefore; there is nothing ABSOLUTE; about any of them; that none of them represents; THE truth; and yet; all or many of them are; extremely useful;". Then in a chapter on "the fourth dimension", to T. C.'s objection "in the physical world; we are also held down by; objective facts; which I still regard as; THE truth", a very clear explanation of dimensions leads to an explanation of the relativity of length while the "interval" is an invariant, and so "a modern scientist no longer speaks of; 'objective facts'; but of; 'invariants under transformations';".

Finally, the suggestion that modern art has various characteristics in common with modern mathematics gives an excuse for the eight modern pictures mentioned above and a summing-up chapter leads to the hope that "Man, with so much; ingenuity and originality; will not let; his social problems; lick him;".

Altogether this is a fascinating, if at times rather irritating, little book, and one to be recommended to all teachers of mathematics as well as to T. C. MITS.

C. O. TUCKEY.

Basic Mathematics for Radio Students. By F. M. COLEBROOK. Pp. x, 270. 10s. 6d. 1946. (Wireless World ; Iliffe)

The contents of this book cover a wide range. Some forty pages are devoted to elementary algebra, " The Fundamental Ideas ". Careful and painstaking explanations are given of the use and meaning of symbols and brackets, and of the processes of multiplication and division. The next twenty pages deal with indices and logarithms, and a further twenty pages are needed for the solution of equations.

The middle one-third (approximately) of the book contains a little more algebra, introducing complex numbers, limits and series, and some elementary geometry and trigonometry, including the concept of the vector.

There remains a section on differential and integral calculus ; and a chapter on " The Applications of Mathematical Ideas to Radio ", in which Kirchhoff's laws, and the characteristics and behaviour of an oscillatory circuit are the chief items. At appropriate places throughout are a few short groups of examples, to which answers are provided.

This catalogue has been given *in extenso*, because it is necessary to raise the question as to whether the range is not too wide. It must be made clear at once that this is not a criticism of the mathematics of any particular part of the book. Also, it appears that the work is based on a series of articles which proved successful in their object of helping engineering students. Yet one must doubt whether many who require the detail of the first one-third of the book could aspire to an understanding of the last one-third without much more assistance than is here given. The author points out that in order to " include in a single work all the main parts of elementary mathematics ", he has chosen " a fairly thorough discussion of the basic ideas " rather than deal with " their detailed development and application ". The discussion is indeed thorough, and the various separate chapters are well done. Yet this means that only Chapter 7 (the last, commencing on page 230) is strictly relevant to that part of the title which says that the book is " for radio students ".

Again, it could well be argued that if a book is designed for radio students, then throughout it should abound in references to radio. This book does not. Apart from an occasional use of such words as ohm and watt, there are only two electrical references of any sort in the first half of its contents, while not until the last chapter are applications of the mathematics really considered. The author justifies his methods by saying that " the really basic ideas of mathematics are common to all its applications ", and Professor G. W. O. Howe, in a foreword to the book, points out that " it would be a great mistake to assume that it is not suited to students of other subjects ".

In short, this volume contains a useful treatment of some selected topics, but students of radio may be disappointed if they obtain it in the hope that it caters for their special needs. F. W. K.

1503. Laurasia in moving from its more or less equatorial position towards the North Pole—where the meridians were *convergent*—became subjected to an *E.-W. compression*, revealed in its orogenies from the Carboniferous onwards and especially in that of the Urals, whereas Gondwana moving towards the equator—where the meridians were *divergent*—became put under *tension in an east-west direction*, which ultimately lead to its fracturing and wide dispersal.—A. L. Du Toit, *Our Wandering Continents*, 1937, p. 305. [Per Mr. S. Melmore.]

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